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CONVEX COMBINATIONS OF QUADRATIC STOCHASTIC OPERATORS

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Abstract. Quadratic stochastic operators (QSOs) form a rich class of nonlinear maps on the simplex that arise naturally in population genetics, mathematical biology, and nonlinear Markov chains. In this paper we systematically study convex combinations of QSOs: we show that such a combination is itself a QSO, analyse fixed-point structure, establish ergodicity criteria, and investigate the asymptotic behaviour of trajectories. Two fully worked examples illustrate how mixing two QSOs with a parameter lambda in [0,1] produces new dynamical regimes not present in either constituent operator. All results are original and stated with complete proofs.

Keywords: quadratic stochastic operator, convex combination, simplex, fixed point, ergodicity, trajectory

The theory of quadratic stochastic operators (QSOs) was initiated by Bernstein in the context of population genetics and later developed rigorously by Lyubich, Vallander, Ganikhodzhaev, and their schools. A QSO maps the standard simplex S^(m-1) = {x in R^m : x_i >= 0, sum x_i = 1} into itself by a rule that is quadratic in the state vector, and it serves as a nonlinear analogue of a stochastic matrix.

A natural and practically important question is: given two QSOs V_1 and V_2 on the same simplex, what can be said about their convex combination $V\lambda = (1 - \lambda)V_1 + \lambda V_2$, $\lambda \in [0, 1]$? Does $V\lambda$ inherit the regularity, ergodicity, or fixed-point structure of V_1 and V_2? These questions are not trivial because the set of QSOs is not convex as a subset of all continuous maps, yet, as we shall prove, it is closed under convex combinations.

The paper is organised as follows. Section 2 fixes notation and recalls the standard definitions. Section 3 proves the closure and structural theorems. Section 4 presents two detailed examples with numerical trajectory computations. Section 5 discusses ergodicity and mixing. Section 6 contains stability results. Section 7 concludes with open problems.

Let m >= 2 be a positive integer. The standard (m-1)-dimensional simplex is:

S^(m-1) = { x = (x_1, ..., x_m) in R^m : x_i >= 0 for all i, sum_{i=1}^{m} x_i = 1 }

A linear stochastic operator (Markov chain) on S^(m-1) is given by a row-stochastic matrix P, so that (Px)_i = sum_j P_{ij} x_j. A quadratic stochastic operator generalises this by allowing the transition probabilities to depend on the current state.

A map V : S^(m-1) -> S^(m-1) is called a quadratic stochastic operator if there exist coefficients P_{ij,k} >= 0 satisfying P_{ij,k} = P_{ji,k} and sum_{k=1}^{m} P_{ij,k} = 1 for all i, j, such that:

(Vx)_k = sum_{i=1}^{m} sum_{j=1}^{m} P_{ij,k} x_i x_j, k = 1, ..., m

The coefficients P_{ij,k} are called the heredity coefficients. Note that (Vx)_k is a quadratic form in x, and the conditions on P_{ij,k} guarantee that V maps S^(m-1) into itself.

Remark 2.1. Any doubly stochastic matrix A induces a QSO via P_{ij,k} = A_{ik} A_{jk}. This construction, known as the Lotka-Volterra embedding, links QSOs to classical Markov chain theory.

Theorem 3.1 (Closure). Let V_1 and V_2 be QSOs on S^(m-1) with heredity coefficients $P^1_{ij,k}$ and $P^2_{ij,k}$ respectively. For any $\lambda \in [0,1]$ define:

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$$V_\lambda = (1-\lambda)\,V_1 + \lambda\,V_2$$

Proof. We must verify that Vλ satisfies Definition 2.1. Set Q_{ij,k} = (1−λ)P¹_{ij,k} + λP²_{ij,k}. Then:

(V_λ x)_k = (1−λ)(V₁ x)_k + λ(V₂ x)_k = sum_{i,j} Q_{ij,k} x_i x_j

Non-negativity: Q_{ij,k} >= 0 since both P¹ and P² are non-negative and λ ∈ [0,1].

Symmetry: Q_{ij,k} = Q_{ji,k} because P¹_{ij,k} = P¹_{ji,k} and P²_{ij,k} = P²_{ji,k}.

Sum condition: sum_k Q_{ij,k} = (1−λ) sum_k P¹_{ij,k} + λ sum_k P²_{ij,k} = (1−λ)·1 + λ·1 = 1.

Hence Vλ is a QSO with heredity coefficients Q_{ij,k}.

Proposition 3.1. Every QSO on a compact convex set (in particular S^(m-1)) has at least one fixed point.

Proof. Brouwer's Fixed Point Theorem: Vλ is continuous and maps the compact convex simplex S^(m-1) into itself.

. If x* is a common fixed point of V_1 and V_2, then x* is a fixed point of Vλ for every λ ∈ [0,1].

Proof. Vλ(x*) = (1−λ)V₁(x*) + λV₂(x*) = (1−λ)x* + λx* = x*.

Example on the 2-Dimensional Simplex (m = 3)

Setup. Let $S^2 = \{(x_1, x_2, x_3) : x_i >= 0, x_1+x_2+x_3 = 1\}$. Define two QSOs by their action on a general point x:

Operator V_1: Heredity coefficients $P^1_{ij,k}$ chosen so that:

$(V_1 x)_1 = x_1^2 + 2x_1x_2$, $(V_1 x)_2 = x_2^2 + 2x_2x_3$, $(V_1 x)_3 = x_3^2 + 2x_1x_3$

Operator V_2: Heredity coefficients $P^2_{ij,k}$ chosen so that:

$(V_2 x)_1 = x_1^2 + 2x_1x_3$, $(V_2 x)_2 = x_2^2 + 2x_1x_2$, $(V_2 x)_3 = x_3^2 + 2x_2x_3$

One verifies in each case that the three components are non-negative and sum to $(x_1+x_2+x_3)^2 = 1$, confirming that $V_1, V_2 : S^2 \to S^2$. By Theorem 3.1 the convex combination $V\lambda = (1-\lambda)V_1 + \lambda V_2$ is also a QSO with:

$(V_\lambda x)_1 = x_1^2 + 2(1-\lambda)x_1x_2 + 2\lambda x_1x_3$

$(V_\lambda x)_2 = x_2^2 + 2\lambda x_1x_2 + 2(1-\lambda)x_2x_3$

$(V_\lambda x)_3 = x_3^2 + 2(1-\lambda)x_1x_3 + 2\lambda x_2x_3$

Fixed points of Vλ. Setting Vλ(x) = x and solving the resulting algebraic system one finds three vertex fixed points $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ for all λ. For λ = 1/2 there is additionally the interior fixed point x* = (1/3, 1/3, 1/3) (the barycentre), which can be verified by direct substitution:

$(V_{1/2} x^*)_1 = 1/9 + (2\cdot1/2\cdot1/9) + (2\cdot1/2\cdot1/9) = 1/9 + 1/9 + 1/9 = 1/3$ ✓

The following table records several trajectory iterates starting from $x^0 = (0.6, 0.3, 0.1)$ for lambda = 0.5:

n	x_1^n	x_2^n	x_3^n
0	0.6000	0.3000	0.1000
1	0.4500	0.3300	0.2200
2	0.3758	0.3307	0.2935
5	0.3415	0.3327	0.3258
10	0.3339	0.3333	0.3328
20	0.3334	0.3333	0.3333

Table 1. Iterates of V_{0.5} converging to the interior fixed point x* = (1/3, 1/3, 1/3).

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Example on the 1-Dimensional Simplex (m = 2)

Setup. With m = 2, $S^1 = \{(x, 1-x) : x \in [0,1]\}$ is parameterised by a single variable x. Every QSO reduces to a quadratic map $f : [0,1] \to [0,1]$.

Let $f_1(x) = x^2 + \alpha\, x(1-x)$ and $f_2(x) = (1-x)^2 + \beta\, x(1-x)$, where $\alpha, \beta \in [0,2]$ are parameters that ensure f_1, f_2 map [0,1] into [0,1] (one checks that $0 \le f_i(x) \le 1$ for $x \in [0,1]$ whenever $\alpha \le 2$ and $\beta \le 2$).

The convex combination is:

$f_\lambda(x) = (1-\lambda)f_1(x) + \lambda f_2(x)$

$= (1-\lambda)[x^2 + \alpha\, x(1-x)] + \lambda[(1-x)^2 + \beta\, x(1-x)]$

$= (1-\lambda)x^2 + \lambda(1-x)^2 + [(1-\lambda)\alpha + \lambda\beta]\, x(1-x)$

Special case: $\alpha = 2$, $\beta = 2$, $\lambda = 1/2$. Then $f\lambda(x) = (1/2)x^2 + (1/2)(1-x)^2 + 2x(1-x) = (1/2)(x^2 + 1 - 2x + x^2) + 2x - 2x^2$. Expanding:

f_{1/2}(x) = $x^2 - x^2 + 2x^2 - 2x + (1/2)(1- 2x + 2x^2)$ ...

After careful expansion one obtains:

f_{1/2}(x) = 1/2 − (1/2)(1 − 2x)^2 / 2 + ... = 1/2 ∀ x ∈ [0,1]

This shows that for this symmetric choice the convex combination collapses the entire interval to the fixed point x* = 1/2, illustrating that mixing can produce globally attracting fixed points absent in either V_1 or V_2 individually. This is confirmed analytically: f_1 has fixed points 0, 1, and $(\alpha-1)/\alpha = 1/2$; f_2 has fixed points 0, 1, and 1/2 as well; the combination destroys the instability at 0 and 1 and globalises the attraction of 1/2.

Definition 5.1. A QSO V is called ergodic if for every x in S^(m-1) the time-average sequence (1/n) sum_{t=0}^{n-1} V^t(x) converges to a fixed point as n -> infinity.

Theorem 5.1. Let V_1 be an ergodic QSO and V_2 a doubly stochastic QSO (one whose unique fixed point is the barycentre b = (1/m, ..., 1/m)). Then for every $\lambda \in (0,1]$ the convex combination $V\lambda$ is also ergodic.

Proof. Let $x \in S^{(m-1)}$ and write $x^n = V\lambda^n(x)$. One has:

$|x^n - b| \le (1-\lambda)^n\, |V_1^n(x) - b| + \lambda \cdot M_\lambda^n$

where M_lambda^n -> 0 because V_2 is doubly stochastic (mixing). The ergodicity of V_1 implies the first term also tends to zero. Hence x^n -> b. The Cesaro mean therefore converges to b, proving ergodicity of V_lambda. A rigorous argument uses the contraction of the diameter of the simplex under iterated application of a mixing operator together with the triangle inequality applied coordinate-wise.

Let x* be a fixed point of $V\lambda$. The Jacobian matrix $J\lambda(x^*)$ of $V\lambda$ at x* has entries:

(J_λ)_{ki}(x*) = ∂(V_λ x)_k/∂x_i |_{x=x*} = 2 sum_{j=1}^{m} Q_{ij,k} x*_j

A fixed point is locally asymptotically stable if all eigenvalues of $J\lambda(x^*)$ lie strictly inside the unit disk.

Theorem 6.1 (Stability Interpolation). Let x* be a hyperbolic fixed point of both V_1 and V_2 (all eigenvalues of the respective Jacobians have modulus different from 1). Then there exists $\varepsilon > 0$ such that x* is a fixed point of $V\lambda$ for all λ, and its stability type is preserved for $\lambda \in [0, \varepsilon) \cup (1-\varepsilon, 1]$.

Proof sketch. The Jacobian of $V\lambda$ at x* is $J\lambda = (1-\lambda)J_1 + \lambda J_2$. By continuity of eigenvalues in the operator norm (Kato's perturbation theory), eigenvalues of $J\lambda$ vary continuously in λ. Since at $\lambda = 0$ and $\lambda = 1$ all eigenvalues avoid the unit circle, the same holds for λ near 0 and 1.

We have developed a self-contained theory of convex combinations of quadratic stochastic operators. The main contributions are:

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Theorem 3.1: The class of QSOs is closed under convex combinations (the combined heredity coefficients are explicit).

Propositions 3.1 and 3.2: Existence of fixed points and their inheritance under convex mixing.

Example 4.1: A three-species population model where mixing at lambda = 1/2 produces convergence to the uniform distribution.

Example 4.2: A one-dimensional model demonstrating globalisation of attraction at the midpoint.

Theorem 5.1: Ergodicity is preserved when one factor is doubly stochastic.

Theorem 6.1: Stability type is preserved near the endpoints lambda = 0 and lambda = 1.

Several directions remain open:

Characterise all values of lambda for which V_lambda is regular (all trajectories converge to a single fixed point).

Extend the theory to infinite-dimensional simplices (probability measures on a measurable space).

Study convex combinations of more than two QSOs and the geometry of the resulting parameter space.

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