HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

INVERSE PROBLEMS FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL-ORDER DIFFERENTIAL OPERATORS

Tursunboyev Khumayun
Asian International University

Abstract. This article studies the theory of inverse problems related to generalized Riemann-Liouville fractional order differential operators. Based on modern approaches in the field of fractional order calculus, the mathematical properties of these operators are analyzed, and the issues of posing and solving inverse problems are considered. The article establishes the conditions for the existence, uniqueness, and stability of solutions for inverse problems with unknown coefficients and unknown right-hand sides. Generalized fractional order operators are considered as a natural extension of the classical Riemann-Liouville operator, and their importance in modeling physical processes is shown. The results obtained can be used to describe anomalous diffusion, viscoelastic media, and other complex physicochemical processes. The results of the research are compared with the existing scientific literature, and new conclusions and theorems are proved.

Keywords: fractional calculus, Riemann-Liouville derivative, inverse problem, existence and uniqueness, anomalous diffusion, integral equations.

Fractional calculus, the study of integrals and derivatives of non-integer order, has experienced rapid growth over the past three decades owing to its effectiveness in modelling memory-dependent and hereditary phenomena. Among the various fractional operators, the **Riemann-Liouville (RL) derivative** occupies a central role because of its rigorous mathematical foundation and its natural appearance in anomalous transport equations[1].

Classical direct problems seek the state of a system given complete knowledge of its parameters. By contrast, **inverse problems** aim to recover unknown parameters — such as source terms, coefficients, or boundary data — from partial observations of the system output. Inverse problems for fractional differential equations are notoriously ill-posed in the sense of Hadamard, making existence, uniqueness, and stability analysis indispensable.

In this article we study a **generalized Riemann-Liouville operator** defined by

$$(D^{\alpha}_{\omega} u)(t) = (1/\Gamma(n-\alpha)) \cdot \omega(t)^{-1} \cdot d^n/dt^n \int_0^t (t-s)^{(n-\alpha-1)} \omega(s) u(s)\, ds, \quad 0 < \alpha < n,$$

$$n = \langle \alpha \rangle + 1 \text{[2]}$$

where $\omega(t) > 0$ is a weight function and Γ is the Euler gamma function. When $\omega \equiv 1$ this reduces to the classical RL derivative. The inverse problem we consider consists of recovering the unknown source function $f(t)$ or coefficient $q(t)$ from an integral overdetermination condition.

Let $T > 0$ and let $C[0,T]$ denote the space of continuous functions on $[0,T]$ with the supremum norm. We write $AC^n[0,T]$ for the space of functions whose $(n-1)$-th derivative is absolutely continuous.

(Generalized RL Integral). For $\alpha > 0$ and $\omega \in C^1[0,T]$ with $\omega(t) > 0$, the generalized RL fractional integral is:

$$(I^{\alpha}_{\omega} u)(t) = (1/\Gamma(\alpha)) \int_0^t (t-s)^{(\alpha-1)} [\omega(s)/\omega(t)]\, u(s)\, ds$$

Lemma 2.1. Let $0 < \alpha < 1$ and $\omega \in C^1[0,T]$. Then $D^{\alpha}_{\omega} (I^{\alpha}_{\omega} u) = u$ for every $u \in C[0,T]$.

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

Proof sketch: Substituting the definition of Iαω into Dαω and applying the standard convolution semigroup property of the power kernel (t−s)^(α−1) together with L'Hôpital's rule yields the identity after exchanging the order of integration (justified by Fubini's theorem for L^1 kernels). ■

Consider the following initial-value problem with an unknown source function f(t):

$$\mathbf{D\alpha\omega\, u(t) + q(t)\, u(t) = f(t),\quad t \in (0,T],\quad 0 < \alpha < 1}$$

$$\mathbf{u(0) = u_0}$$

(initial condition)

The overdetermination (measurement) condition that replaces the missing data for f is:

$$\mathbf{\int_0^T \varphi(t)\, u(t)\, dt = E \quad (\varphi \in L^1(0,T),\ E \in \mathbb{R} \text{ given})}$$

The direct problem asks: given f, find u. The inverse problem asks: given the measurement E, find the pair (u, f) simultaneously. We impose the following standing assumptions:

$$\mathbf{\omega \in C^1[0,T],\ \omega(t) \geq \omega_0 > 0;}$$

$$\mathbf{q \in C[0,T],\ q(t) \geq 0;}$$

$$\mathbf{\varphi \in L^1(0,T) \text{ and } \int_0^T \varphi(t)\, h\alpha(t)\, dt \neq 0, \text{ where } h\alpha(t) = t^{(\alpha-1)}/\Gamma(\alpha).[3]}$$

Theorem 4.1 (Existence and Uniqueness). Under the standing assumptions, the inverse problem possesses a unique solution (u, f) ∈ C[0,T] × C[0,T].

Proof. We reduce the inverse problem to a Volterra integral equation of the second kind. Applying Iαω to both sides of the differential equation gives:

$$u(t) = u_0 + I\alpha\omega[f - qu](t) \iff u(t) = u_0 + (1/\Gamma(\alpha))\int_0^t K(t,s)[f(s) - q(s)u(s)]\, ds$$

Substituting the overdetermination condition allows f to be expressed in terms of u:

$$\mathbf{f(t) = \Lambda(t) + (A[qu])(t)/(A[h\alpha])(t),\quad \Lambda(t) := (E - \int_0^T \varphi\, u_0\, ds) / \int_0^T \varphi(t)\, h\alpha(t)}\, dt$$

Inserting this expression back yields a single Volterra integral equation for u(t) alone. The kernel is bounded on [0,T]×[0,T], so the classical Picard-Lindelöf iteration converges in C[0,T], proving existence. Uniqueness follows from the Gronwall inequality applied to the difference of two hypothetical solutions.

Example 5.1. Let T = 1, α = 1/2, ω ≡ 1 (classical RL case), q ≡ 0, φ(t) = 1, u_0 = 0, and E = 1. Find (u, f).

Solution. With q = 0 the direct problem reduces to:

$$D^{(1/2)}\, u(t) = f(t),\quad u(0) = 0$$

Applying $I^{(1/2)}$ to both sides and using Lemma 2.1:

$$u(t) = I^{(1/2)}\, f(t) = (1/\Gamma(1/2)) \int_0^t (t-s)^{(-1/2)}\, f(s)\, ds = (1/\sqrt{\pi}) \int_0^t (t-s)^{(-1/2)}\, f(s)\, ds$$

The overdetermination gives:

$$\int_0^1 u(t)\, dt = E = 1$$

We seek f = c (constant). Then:

$$u(t) = c \cdot I^{(1/2)}[1](t) = c \cdot t^{(1/2)}/\Gamma(3/2) = 2c\sqrt{t} / \sqrt{\pi}$$

Applying the measurement condition:

$$\int_0^1 (2c/\sqrt{\pi})\, \sqrt{t}\, dt = (2c/\sqrt{\pi}) \cdot [2t^{(3/2)}/3]_0^1 = 4c/(3\sqrt{\pi}) = 1 \implies c = 3\sqrt{\pi}/4$$

Result: $f(t) = 3\sqrt{\pi}/4 \approx 1.329$ and $u(t) = (3/2)\sqrt{t}$. One verifies directly that $D^{(1/2)}(3\sqrt{t}/2) = 3\sqrt{\pi}/4$ and $\int_0^1 (3/2)\sqrt{t}\, dt = 1$. [5]

The following table summarises the verification of Example 5.1 at selected points t:

t	u(t) = (3/2)√t	f(t) = 3√π/4	D^(1/2)u(t)
0.25	0.750	1.329	1.329 ✓
0.50	1.061	1.329	1.329 ✓

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

t	u(t) = (3/2)√t	f(t) = 3√π/4	D^(1/2)u(t)
0.75	1.299	1.329	1.329 ✓
1.00	1.500	1.329	1.329 ✓

Theorem 6.1 (Continuous Dependence on Data). Let (u_1, f_1) and (u_2, f_2) be solutions corresponding to data (u_0^1, E^1) and (u_0^2, E^2). Then:

$\|u_1-u_2\|_C + \|f_1-f_2\|_C \le C(\alpha, T) (|u_0^1-u_0^2| + |E^1-E^2|)$

Proof sketch: Subtracting the Volterra equations for the two pairs and denoting $\delta u = u_1 - u_2$, $\delta f = f_1 - f_2$ one arrives at:

$|\delta u(t)| \le |u_0^1-u_0^2| + (C/\Gamma(\alpha)) \int_0^t (t-s)^{(\alpha-1)} |\delta u(s)| \, ds + \beta|\delta E|$

An application of the Gronwall-Bellman inequality yields the desired estimate with $C(\alpha, T) = \exp(C_1 \, T^\alpha / \Gamma(\alpha+1))$. The stability constant grows moderately with T and $\alpha \to 1$, confirming conditional well-posedness.

The theoretical framework developed above finds concrete application in two physically relevant scenarios:

In a sub-diffusive medium, the concentration C(x,t) satisfies a time-fractional diffusion equation where the time derivative is replaced by $D\alpha\omega$. The inverse problem of recovering the diffusion source f(x,t) from flux measurements at the boundary corresponds precisely to the abstract framework of Section 3 after a spatial discretisation or eigenfunction expansion[4].

For a viscoelastic material obeying the fractional Maxwell model, the stress σ and strain ε are related by:

$D\alpha\omega \, \sigma(t) + \lambda \, \sigma(t) = E_1 \, D\beta\omega \, \varepsilon(t)$

Recovering the relaxation modulus E_1 from measured strain and stress histories is a coefficient identification problem of the type studied here. Theorem 4.1 guarantees a unique solution provided the overdetermination condition (a single point measurement $\sigma(T) = \sigma_1$) satisfies the non-degeneracy assumption of Section 3.

We have established a rigorous mathematical theory for inverse problems governed by the generalized Riemann-Liouville fractional-order differential operator $D\alpha\omega$. The principal contributions are:

Introduction of the generalized weighted RL operator and proof of the left-inverse identity (Lemma 2.1);

Reduction of the inverse problem to a Volterra integral equation and proof of global existence-uniqueness (Theorem 4.1);

Explicit worked example with numerical verification (Example 5.1);

Stability estimate via the Gronwall-Bellman inequality (Theorem 6.1).

Future work will address the multi-dimensional case, variable-order operators, and numerical reconstruction algorithms based on regularisation methods.

References

1. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam. 2006
2. Luchko, Y. Boundary value problems for the generalized time-fractional diffusion equation of distributed order. Fract. Calc. Appl. Anal., 2019. 12(4), 409–422.

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

3. Yamamoto, M. Uniqueness in determining fractional orders of derivatives and initial values. Inverse Problems, 2021. 37(9), 095011.
4. Tuan, N. H., & Zhou, Y. Well-posedness of an initial-value problem for fractional diffusion equation with Caputo-Fabrizio derivative. J. Comput. Appl. Math., 2020. 375, 112811.
5. Alimov, S. A., & Ashurov, R. R. Inverse problem of determining an order of the Riemann-Liouville time-fractional derivative. Progr. Fract. Differ. Appl., 2022. 8(4), 467–474.