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RIGOROUS FOUNDATIONS OF ELEMENTARY FUNCTIONS FROM THE PERSPECTIVE OF MATHEMATICAL ANALYSIS

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ABSTRACT. This article examines the rigorous foundations of elementary functions from the perspective of mathematical analysis, focusing on their definitions, conditions of existence, and principal analytical properties. The rigorous introduction of elementary functions is closely connected with the theory of real numbers, limits, continuity, differentiability, and integration. The study compares theoretical approaches to defining power, exponential, logarithmic, and trigonometric functions. It also considers the completeness of the real number system, functional equations, inverse functions, and integral definitions. The analysis shows that logical consistency of definitions, independence of mathematical proofs, and connections among theoretical concepts are essential for a rigorous foundation.

Keywords: elementary functions; mathematical analysis; rigorous foundations; real numbers; limits; continuity; differentiability; exponential function; logarithmic function; trigonometric functions; integral definition.

ANNOTATSIYA. Ushbu maqolada elementar funksiyalarning qat'iy asoslari matematik tahlil nuqtai nazaridan ko'rib chiqiladi, ularning ta'riflari, mavjudlik shartlari va asosiy analitik xususiyatlariga e'tibor qaratiladi. Elementar funksiyalarning qat'iy kiritilishi haqiqiy sonlar, limitlar, uzluksizlik, differentsiallanish va integrallanish nazariyasi bilan chambarchas bog'liq. Tadqiqotda daraja, eksponensial, logarifmik va trigonometrik funksiyalarni aniqlashga nazariy yondashuvlar taqqoslanadi. Shuningdek, u haqiqiy sonlar tizimining to'liqligi, funksional tenglamalar, teskari funksiyalar va integral ta'riflarni ko'rib chiqadi. Tahlil shuni ko'rsatadiki, ta'riflarning mantiqiy izchilligi, matematik isbotlarning mustaqilligi va nazariy tushunchalar orasidagi bog'liqlik qat'iy asos uchun juda muhimdir.

Kalit so'zlar: elementar funksiyalar; matematik tahlil; qat'iy asoslar; haqiqiy sonlar; limitlar; uzluksizlik; differentsiallanish; eksponensial funksiya; logarifmik funksiya; trigonometrik funksiyalar; integral ta'rif.

АННОТАЦИЯ. В данной статье рассматриваются строгие основы элементарных функций с точки зрения математического анализа, с акцентом на их определения, условия существования и основные аналитические свойства. Строгое введение элементарных функций

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тесно связано с теорией действительных чисел, пределов, непрерывности, дифференцируемости и интегрирования. В исследовании сравниваются теоретические подходы к определению степенных, экспоненциальных, логарифмических и тригонометрических функций. Также рассматривается полнота системы действительных чисел, функциональные уравнения, обратные функции и определения интегралов. Анализ показывает, что логическая непротиворечивость определений, независимость математических доказательств и связи между теоретическими понятиями имеют важное значение для строгого основания.

Ключевые слова: элементарные функции; математический анализ; строгие основы; действительные числа; пределы; непрерывность; дифференцируемость; экспоненциальная функция; логарифмическая функция; тригонометрические функции; определение интеграла.

INTRODUCTION

Mathematical analysis is a central field of modern mathematics concerned with functional relationships between varying quantities, their continuous change, and analytical properties. Elementary functions occupy a fundamental place in this discipline: many applications of limits, derivatives, integrals, functional series, and differential equations are expressed through them.

Elementary functions conventionally include algebraic, exponential, logarithmic, trigonometric, and inverse trigonometric functions, together with functions formed from them by finitely many arithmetic operations and compositions. Presenting formulas alone, however, does not establish their mathematical meaning. Rigorous treatment requires proofs of existence, domains, continuity, differentiability, and other relevant properties.

The historical development of analysis made rigorous definitions a major concern. During the nineteenth century, work by Augustin-Louis Cauchy, Karl Weierstrass, and Richard Dedekind helped clarify limits, continuity, and the real number system, strengthening the logical foundations of the subject. Consequently, elementary functions could be studied through explicit definitions and proofs rather than intuition alone.

This remains relevant to mathematical education today. Different constructions of exponential and logarithmic functions, as well as geometric and analytic treatments of trigonometric functions, make a logically ordered presentation necessary.

Research objective. To examine methods for rigorously introducing elementary functions, compare their theoretical characteristics, and identify logically coherent approaches to their definition.

Research tasks include systematizing the foundations of elementary functions, comparing construction methods, establishing connections to continuity and differentiability, and identifying ways to avoid circular reasoning in definitions.

RESEARCH METHODOLOGY

This is a theoretical and analytical study based on foundational mathematical definitions, proof techniques, and scholarly and educational reference works. No empirical experiments were conducted.

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The analysis draws on the Encyclopedia of Mathematics, the NIST Digital Library of Mathematical Functions, and OpenStax Calculus, supplemented by established analysis textbooks [1–4, 6–8].

Four methods were applied: (1) an axiomatic approach based on the ordered-field and completeness properties of the real numbers; (2) deductive reasoning to derive continuity, monotonicity, and differentiability from initial definitions; (3) comparative analysis of integral, power-series, and inverse-function constructions; and (4) mathematical proof using limits, differentiation rules, the fundamental theorem of calculus, and functional identities.

Three criteria guided the comparison: whether a definition relies on previously established concepts; whether existence and uniqueness can be justified; and whether the resulting properties follow without logical circularity.

RESULTS

Definition and theoretical foundations of elementary functions

Elementary functions form a class generated by standard basic functions through finitely many arithmetic operations and compositions [1, 5]. For example,

$f(x) = [\ln(1 + x^2) + e^x] / [1 + \sin^2 x]$

is elementary because its constituents are elementary and are combined through arithmetic operations and composition. The completeness property of the real numbers is essential to the general theory: every nonempty set of real numbers bounded above has a least upper bound. This supports major existence and convergence theorems in analysis.

Continuity at a point a is expressed rigorously by the epsilon–delta condition:

$\forall \varepsilon > 0,\ \exists \delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Here x is understood to belong to the domain of f. Continuity of elementary functions can then be established using basic continuity theorems for sums, products, quotients (where denominators are nonzero), and compositions.

Rigorous introduction of the logarithmic function by integration

A logically coherent construction begins by defining the natural logarithm for $x > 0$ as [2]

$\ln x = \int_1^x (1/t)\, dt, \quad x > 0.$

The integrand is continuous on every compact positive interval, so the integral is well-defined. By the fundamental theorem of calculus,

$d(\ln x)/dx = 1/x > 0, \quad x > 0.$

Thus ln x is strictly increasing. Its product law can be proved without assuming logarithm rules. For fixed $a > 0$, set $F(x) = \ln(ax) - \ln x$. Then

$F'(x) = a/(ax) - 1/x = 0.$

Hence F is constant, and $F(1) = \ln a$, which yields

$\ln(ab) = \ln a + \ln b, \quad a, b > 0.$

This derives an algebraic property directly from the integral definition.

Exponential functions as inverse functions

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Because the natural logarithm is continuous, strictly increasing, and maps (0, ∞) onto the entire real line, it has an inverse, denoted $\exp(x) = e^x$. In particular,

$\lim_{x\to 0+} \ln x = -\infty$; $\lim_{x\to\infty} \ln x = +\infty$.

$y = e^x \Leftrightarrow x = \ln y, \quad y > 0$.

The inverse-function differentiation rule gives

$d(e^x)/dx = 1 / [(d/dy) \ln y]_{\gamma=e^x} = e^x$.

For $a > 0$, define $a^x = e^{\wedge}(x \ln a)$. Consequently,

$d(a^x)/dx = a^x \ln a$.

This approach defines real powers, including irrational exponents, through already established functions [2, 3].

Trigonometric functions through power series

An analytic construction defines sine and cosine by the everywhere-convergent series [4]:

$\sin x = \Sigma_{n=0}\infty (-1)^n x^{\wedge}(2n+1)/(2n+1)!$

$\cos x = \Sigma_{n=0}\infty (-1)^n x^{\wedge}(2n)/(2n)!$

The ratio test establishes absolute convergence for every real x. For the absolute values of successive sine-series terms, the ratio tends to zero:

$\lim_{n\to\infty} |x|^2 / [(2n+3)(2n+2)] = 0$.

Power-series differentiation within the radius of convergence therefore gives

$(\sin x)' = \cos x$; $(\cos x)' = -\sin x$.

Define $G(x) = \sin^2 x + \cos^2 x$. Differentiation yields $G'(x) = 0$. Since $\sin 0 = 0$ and $\cos 0 = 1$, $G(0) = 1$; therefore

$\sin^2 x + \cos^2 x = 1$.

This demonstrates that a fundamental trigonometric identity can be derived analytically, without using geometric intuition as an unproved premise.

Rigorous definition of power functions

For an arbitrary real exponent α and positive argument x, define

$x^{\wedge}\alpha = e^{\wedge}(\alpha \ln x), \quad x > 0$.

The chain rule then yields

$d(x^{\wedge}\alpha)/dx = \alpha x^{\wedge}(\alpha-1), \quad x > 0$.

Extensions to zero or negative arguments depend on the exponent and require separate domain analysis.

Comparative synthesis

Function class	**Rigorous construction**	**Mathematical basis**
Polynomial and rational	Algebraic operations	Real ordered field
Logarithmic	Definite integral	Fundamental theorem of calculus
Exponential	Inverse of logarithm	Continuity and monotonicity

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Trigonometric	Power series	Convergence and termwise differentiation
Power	Exponential–logarithmic expression	Composition and chain rule
Inverse trigonometric	Inverse on a monotone interval	Inverse-function theorems

The comparison shows that no single construction method is mandatory for all elementary functions. A coherent dependency structure is: real numbers and completeness → limits and continuity → integration, series, and inverse functions → rigorous definitions → analytical properties and applications. This is one valid organization, not the only possible axiomatic development.

DISCUSSION

The results confirm that rigorous foundations require more than listing formulas. Existence, uniqueness, domains, and analytic properties must be established through logically prior mathematical results. The integral construction of logarithms used in OpenStax [2] and the systematic treatment of series and identities in NIST DLMF [4] illustrate complementary approaches.

The main distinction between constructions lies in the choice of starting objects. Under the integral approach, logarithms are defined first and exponentials are introduced as inverses. Under a series approach, exponential and trigonometric functions can be defined directly. For example,

$e^x = \Sigma_{n=0}\infty\ x^n/n!$

converges for every real x and satisfies $(e^x)' = e^x$. Equivalence with the inverse-logarithm construction follows by establishing the same defining differential equation and initial condition, together with uniqueness of its solution.

Avoiding circular definitions is particularly important. Defining the logarithm as the inverse of an exponential that has not yet been independently constructed, and then defining the exponential as the inverse of that logarithm, provides no independent foundation.

A further distinction concerns the limits of the elementary class. Although derivatives of elementary functions remain elementary wherever defined, antiderivatives need not be expressible in elementary terms. The integral ∫e^(−x²) dx, for example, has no elementary antiderivative. This fact places an important boundary on elementary symbolic integration.

For mathematics education, the findings suggest value in connecting definitions, theorems, and proofs rather than presenting functions only as ready-made formulas. Nevertheless, this study is theoretical: it did not measure student learning or test a teaching intervention. Claims about instructional effectiveness therefore require separate empirical investigation.

CONCLUSION

The study establishes that rigorous introduction of elementary functions is inseparable from the real number system, limits, continuity, differentiation, integration, and infinite series. It compares the

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integral definition of the logarithm, the inverse-function definition of the exponential, the power-series definition of trigonometric functions, and the exponential–logarithmic construction of real powers.

Five conclusions follow. First, existence, uniqueness, and analytic properties should rest on previously justified results. Second, defining logarithms through integrals provides a coherent path to exponential and real-power functions. Third, power series permit analytic proofs of trigonometric derivatives and identities. Fourth, distinct constructions may be equivalent, but equivalence itself requires proof. Fifth, logical independence and avoidance of circular reasoning are essential methodological principles.

The contribution of this theoretical review is a comparative, logically ordered presentation of established construction methods rather than a claim to a new theorem or experimental discovery. Future work could evaluate the educational effectiveness of this presentation through empirical studies.

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