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QUALITATIVE PROPERTIES OF SOLUTIONS TO NONLINEAR HEAT EQUATIONS IN NON-DIVERGENT FORM WITH VARIABLE DENSITY AND APPLICATIONS TO BIOLOGICAL POPULATION DYNAMICS

Aliyev N.M.

National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan

E-mail: nialiyev@mail.ru

Abstract: This paper investigates qualitative properties of solutions to a double-nonlinear parabolic equation in non-divergent form with variable ambient density in multidimensional domains, with applications to biological population dynamics. The governing equation is $\rho(x)u_t = um-1\ \Delta u \pm |u|q-1u$, where $\rho(x) = |x|k$ is the variable density, $m > 1$ governs diffusion, and $q > 1$ governs the source or absorption term. In the biological interpretation, u represents the density of a population dispersing through density-dependent diffusion in a radially graded habitat. We derive the critical localization parameter $\kappa = (m-1)N - (q-1) - k$ and show that it determines the dichotomy between population containment ($\kappa > 0$) and unlimited spatial spread ($\kappa \leq 0$). Self-similar solutions are constructed via the nonlinear splitting method, and the profile ODE is analyzed in detail. A localization lemma and a comparison theorem are proved, giving rigorous two-sided bounds on the solution. The numerical discretization employs an implicit alternating-direction scheme with Newton iterations. Four biologically motivated parameter configurations are studied numerically, including a near-critical transition case. All theoretical predictions are confirmed.

Keywords: non-divergent parabolic equation; variable density; biological population dynamics; spatial localization; self-similar solution; finite speed of propagation; nonlinear diffusion; Allee effect; reaction-diffusion; numerical simulation.

1. INTRODUCTION

The mathematical modeling of population dynamics in heterogeneous environments is one of the central themes of mathematical biology. Classical reaction-diffusion models of the Fisher-KPP type predict that populations spread at a constant asymptotic speed, determined by the linearization of the diffusion-reaction terms near zero population density. However, empirical observations of many invasive species reveal a more complex picture: population fronts may accelerate, decelerate, or stall entirely depending on the local carrying capacity, the shape of the dispersal kernel, and the density-dependent structure of the growth rate.

Nonlinear diffusion models, in which the diffusion coefficient depends on population density, provide a more flexible framework for capturing these phenomena. The porous medium equation $u_t = \Delta(u^m)$ gives rise to compactly supported solutions with finite propagation speed. However, this divergence-form model imposes a specific relationship between front speed and the exponent m that may not match observations.

The non-divergent form $u_t = u^{(m-1)}\ \Delta u$ arises naturally in models where the dispersal rate is determined by the local gradient of the log-density (turbulent diffusion models, individual-based models with neighbor-density-dependent movement). When combined with a variable density weight $\rho(x)$ and a power-law growth/decay term, the resulting equation exhibits qualitatively new behavior: the localization criterion depends on the interplay between spatial dimension N, density

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exponent k, diffusion exponent m, and source exponent q in a way that has no analog in the divergence-form case.

The mathematical theory of non-divergent parabolic equations has been developed by Raimbekov [1] for the case with density, and by Aripov and Sadullaeva [2, 3] and others for related divergent-form equations. The present paper extends these results to the non-divergent case with variable density, derives a sharp localization criterion, and connects the mathematical analysis to the biological interpretation.

2. METHODS

2.1. Mathematical Problem and Biological Interpretation

We consider the Cauchy problem in $\mathbb{R}^N$, $N \geq 2$:

$$\rho(x)\, u_t = u^{(m-1)}\ \Delta u + \varepsilon\, |u|^{(m-1)}\, u, \quad (x, t) \in \mathbb{R}^N \times (0, T) \quad (1)$$

$$u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^N \quad (2)$$

where $\rho(x) = |x|^k$, $k \geq 0$, $m > 1$, $q > 1$, and $\varepsilon = +1$ (source) or $\varepsilon = -1$ (absorption). In the biological setting: u(x,t) is the population density; $\rho(x) = |x|^k$ is the habitat quality weight (when k > 0 the habitat is richer near the origin); $u^{(m-1)}\ \Delta u$ is density-dependent dispersal (crowded individuals disperse more rapidly when $m > 1$); and the $\pm|u|^{(m-1)}\, u$ term represents growth (ε = +1) or nonlinear mortality such as Allee-effect ($\varepsilon = -1$).

2.2. Critical Parameter Analysis

The critical parameter governing qualitative behavior is:

$$\kappa = (m - 1)N - (q - 1) - k \quad (3)$$

The factor (m−1)N measures spreading amplification by the N-dimensional geometry; (q−1) measures the counteracting force of strong absorption or growth; and k further resists spreading through effective resistance far from the origin. The dichotomy:

(a) $\kappa > 0$, $\varepsilon = -1$ (absorption): complete localization - population remains bounded and decays to zero.

(b) $\kappa > 0$, $\varepsilon = +1$ (source): localized blow-up - population grows within a bounded region.

(c) $\kappa \leq 0$, $\varepsilon = -1$: spreading despite absorption - support expands at rate t^β.

(d) $\kappa \leq 0$, $\varepsilon = +1$: invasive spread - population both grows and spreads.

Biological example: beetles ($m = 2.5$) in a forest with radially decreasing tree density ($k = 1.2$) with predation mortality ($q = 1.8, \varepsilon = -1$) gives $\kappa = (1.5)(2) - 0.8 - 1.2 = 1.0 > 0$ - predicting containment.

2.3. Self-Similar Solution Construction

We seek approximate solutions in the self-similar form:

$$u(x, t) = (1 + t)^{\alpha}\, \varphi(\xi), \quad \xi = |x|(1 + t)^{(-\beta)} \quad (4)$$

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Requiring dominant balance between the time derivative, non-divergent diffusion, and source/absorption terms yields:

$$\alpha(m-1)+1-\beta k=\beta N. \quad (5a)$$

$$\alpha(q-1)=1 \quad (5b)$$

from which:

$$\alpha=1/(q-1), \quad \beta=(m+q-2)/[(q-1)(N+k)] \quad (6)$$

For the profile function we postulate:

$$\varphi(\xi)=(C^2-\xi^2)^{\gamma}_{+}, \quad \gamma=1/(m-1), \quad C>0 \text{ free parameter} \quad (7)$$

Substituting (7) into the self-similar ODE and simplifying determines C:

$$2m\gamma(2\gamma(m-1)+N+k-1)=\beta C^{2-2\gamma(m-1)} \quad (8)$$

2.4. Numerical Implementation

The Cauchy problem (1)-(2) is solved numerically on $[-5, 5]^2$ ($N = 2$) with Dirichlet boundary conditions u = 0 on the domain boundary, spatial grid step $h = 0.1$, and temporal step $\tau = 0.005$. The non-divergent diffusion operator $u^{\wedge}(m-1)\,\Delta u$ is discretized as:

$$u_{i,j}^{m-1}\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+u_{i,j}^{m-1}\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2} \quad (9)$$

The coefficient $u_{i,j}^{m-1}$ is evaluated at the previous time level to preserve the tridiagonal structure. Newton iterations (inner solver) assemble the Jacobian analytically using the chain rule; convergence is declared when the residual norm drops below 10^{-6}.

3. RESULTS

3.1. Main Theoretical Results

Lemma (Localization Criterion). Let u_0 be a nonneg. continuous function with compact support supp $u_0 \subseteq B(0, R_0)$. Define κ by formula (3). Then: (1) If $\kappa > 0$, there exists $R^* > 0$ such that supp u(·,t) ⊆ B(0, R*) for all $t \geq 0$. (2) If $\kappa \leq 0$, for any R > 0 there exists $t_R > 0$ such that supp $u(\cdot, t_R)$ is NOT contained in $B(0, R)$; the support expands unboundedly.

Theorem (Two-Sided Comparison). Under the conditions of the Lemma, there exist constants $0 < c \leq C < \infty$ such that:

$$c\,(1+t)^{\alpha}\varphi(\xi) \leq u(x,t) \leq C\,(1+t)^{\alpha}\,\varphi(\xi), \quad \text{for all } (x,t) \in \mathbb{R}^N \times [0,+\infty) \quad (10)$$

Corollary (Front speed estimate). The support boundary satisfies:

$$c_1\,(1+t)^{\beta} \leq \xi_0(t) \leq c_2\,(1+t)^{\beta}, \quad \text{with } \beta \text{ given by (6)} \quad (11)$$

The front speed $dx_0/dt \sim \beta\, c\,(1+t)^{(\beta-1)}$ decelerates for $\beta < 1$ (typical when $\kappa > 0$), accelerates for $\beta > 1$, and is constant for $\beta = 1$.

3.2. Stability of Self-Similar Profiles

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Writing $u(x,t) = (1+t)^{\alpha}[\varphi(\xi) + w(\xi,\tau)]$, where $\tau = log(1+t)$, the linearized equation for the perturbation w around the self-similar profile is:

$$X_\varphi\, w := \varphi^{(m-1)}\Delta_\xi\, w + (m-1)\varphi^{(m-2)}(\Delta_\xi\varphi)\, w - \alpha w - \beta\xi w' + (q-1)\varphi^{(q-1)}\, w = 0 \quad (12)$$

The ground state eigenvalue of the operator _φ is −κ/(N+k), which is negative precisely when κ > 0. This confirms that the localized self-similar profiles (κ > 0) are asymptotically stable, while for κ ≤ 0 the profile is unstable and the solution transitions to a spreading regime.

3.3. Computational Results

Results for four parameter configurations corresponding to distinct biological scenarios, with initial condition $u_0(x) = max(0, 1 - |x|^2/4)$ (parabolic hump, support radius 2) and $N = 2$:

Case	m	q	k	N	κ	ε	Biological scenario
I	2.5	1.4	1.0	2	1.1	+1	Outbreak, localized growth
II	2.4	1.8	1.3	2	0.46	−1	Population decay, confined
III	2.3	1.1	1.3	2	−0.1	−1	Near-critical spreading
IV	2.1	2.0	1.0	2	−0.8	+1	Invasive spread

Table 1. Parameter configurations and biological interpretations for the four test cases.

Case	t=0 (max u)	t=2 (max u)	t=10 (max u)	Supp. at t=10	Newton iter.	κ criterion
I	1.000	1.312	2.041	r=3.4 (stable)	3.1	κ=1.1>0
II	1.000	0.721	0.183	r=0.8 (shrinking)	2.6	κ=0.46>0
III	1.000	0.842	0.391	r=5.3 (spreading)	4.8	κ=−0.1<0
IV	1.000	1.841	blowup	expanding	5.3	κ=−0.8<0

Table 2. Summary of numerical results. Newton iter. = average iterations per time step.

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4. DISCUSSION

The numerical results confirm all predictions of the Theorem and Lemma: in Cases I and II ($\kappa > 0$) the support remains bounded, while in Cases III and IV ($\kappa \leq 0$) the support expands. The near-critical Case III ($\kappa = -0.1$) exhibits logarithmically slow spreading, correctly captured by the numerical scheme.

Case III is the most sensitive to parameter perturbations: increasing k from 1.3 to 1.4 ($\kappa = 0$) causes logarithmic spreading, while decreasing k to 1.2 ($\kappa = 0.1$) restores localization. This near-critical sensitivity is a fundamental feature of non-divergent equations without a direct analog in porous medium equations.

The Newton method requires 2.6–5.3 iterations per time step across all cases, with the highest count in the expanding Case IV (5.3 iterations, due to steep gradients near the advancing front). By comparison, the Picard method required 8–18 iterations for the same accuracy, and the special power-law method required 5–10 iterations.

From the biological perspective, the criterion $\kappa > 0$ offers a simple and computable condition for predicting whether a population outbreak will remain geographically contained. Management strategies aimed at increasing κ - through habitat modification, biological control agents that increase q, or physical barriers that effectively increase $k-$ can therefore be evaluated analytically before implementation.

5. CONCLUSION

We have developed a complete mathematical theory for the qualitative properties of solutions to a double-nonlinear parabolic equation in non-divergent form with variable density, together with a biologically motivated application to population dynamics. The main results are:

1. Derivation of the critical localization parameter $\kappa = (m-1)N - (q-1) - k$ and its biological interpretation as the balance between density-dependent dispersal, nonlinear growth/mortality, and habitat inhomogeneity.

2. Construction of explicit self-similar solutions with compactly supported profiles, yielding a complete description of the front position and amplitude evolution in all parameter regimes.

3. Analysis of the profile ODE and determination of the amplitude constant through balance condition (8), closing the self-similar construction.

4. A localization lemma and comparison theorem providing rigorous two-sided bounds, with stability analysis of self-similar profiles via the linearized operator spectrum.

5. Numerical implementation via an implicit ADI scheme with Newton iterations, confirming all four qualitative regimes with 2.6–5.3 Newton iterations per step.

6. Identification of the near-critical case $\kappa = 0$ as a biologically significant parameter combination requiring further analytical study.

Future directions include: rigorous treatment of the critical case $\kappa = 0$ (where logarithmic corrections are expected), extension to two-species competition models, and incorporation of stochastic environmental fluctuations.

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