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COMPARATIVE ANALYSIS OF THE D'ALEMBERT, CAUCHY, RAABE, AND GAUSS TESTS FOR SERIES WITH POSITIVE TERMS

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Abstract. This article presents a comparative mathematical analysis of the d'Alembert ratio test, Cauchy root test, Raabe test, and Gauss test for determining the convergence or divergence of infinite series with positive terms. The study investigates the theoretical foundations, mathematical conditions, analytical capabilities, and limitations of these classical convergence criteria. Particular attention is devoted to their effectiveness in examining series with exponential, polynomial, logarithmic, and asymptotically defined terms. The research employs theoretical analysis, asymptotic expansion, comparative mathematical reasoning, and representative examples to identify the conditions under which each test provides a definitive conclusion. The results demonstrate that the d'Alembert and Cauchy tests are particularly effective for series exhibiting exponential decay or growth, whereas the Raabe and Gauss tests provide more refined criteria for series whose consecutive-term ratios approach unity. The analysis also establishes that the Gauss test can distinguish certain borderline cases in which the Raabe test remains inconclusive.

Keywords: infinite series, positive-term series, convergence, divergence, d'Alembert ratio test, Cauchy root test, Raabe test, Gauss test, asymptotic analysis, mathematical criteria.

Introduction. Infinite series constitute one of the fundamental concepts of mathematical analysis and play an essential role in calculus, differential equations, approximation theory, numerical analysis, probability theory, and mathematical physics. The convergence of infinite series determines whether the sum of infinitely many terms approaches a finite limit, thereby providing a theoretical foundation for numerous analytical and computational methods.

Consider an infinite series of positive real numbers:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots , \qquad a_n > 0.$$

The sequence of partial sums is defined by

$$S_n = \sum_{k=1}^{n} a_k .$$

Since every term is positive,

$$S_{n+1} = S_n + a_{n+1} > S_n .$$

Consequently, the sequence of partial sums is strictly increasing. The series converges if and only if the sequence $\{S_n\}$ is bounded above.

Although this characterization is theoretically straightforward, establishing boundedness directly is often difficult. For this reason, mathematicians have developed various convergence

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criteria that make it possible to determine the behavior of a series by examining its general term or the relationship between consecutive terms.

Among these criteria, the d'Alembert, Cauchy, Raabe, and Gauss tests are particularly significant because they provide different levels of analytical precision.

The d'Alembert ratio test examines the limiting behavior of the quotient of consecutive terms. The Cauchy root test evaluates the limiting behavior of the n-th root of the general term. The Raabe test refines the ratio test by examining the first-order deviation of the consecutive-term ratio from unity. The Gauss test introduces a more detailed asymptotic representation and can resolve certain critical cases in which simpler criteria fail. Despite their common objective, these tests differ substantially in their assumptions, applicability, and ability to identify convergence.

A central problem in the theory of infinite series is the selection of an appropriate convergence test for a given positive-term series.

A test that is effective for one class of series may provide no information for another. For example, the d'Alembert ratio test is effective for factorial and exponential expressions but is generally inconclusive for ordinary power series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Similarly, the Raabe test can distinguish convergence and divergence for many power-type series but becomes inconclusive at certain critical parameter values.

The principal objective of this study is to compare the theoretical and practical effectiveness of the d'Alembert, Cauchy, Raabe, and Gauss convergence tests.

Specifically, the research seeks to establish their mathematical conditions, compare their analytical strength, identify representative cases of applicability, investigate critical and inconclusive cases, and formulate a rational procedure for selecting an appropriate criterion.

Materials and methods. The research is based on theoretical mathematical analysis and the comparative examination of classical convergence criteria.

The methodological framework includes deductive reasoning, asymptotic expansions, limit calculations, comparison with benchmark series, and the construction of illustrative examples.

$$\sum_{n=1}^{\infty} a_n, \qquad a_n>0.$$

Suppose that the following limit exists:

$$L=\lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

The d'Alembert criterion states that

$$\begin{cases} L<1 & \text{the series converges,} \\ L>1 & \text{the series diverges,} \\ L=1 & \text{the test is inconclusive.} \end{cases}$$

If $L<1$, there exists a number q satisfying

$$L<q<1$$

such that, for sufficiently large n,

$$a_{n+1}\le qa_n.$$

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Repeated application yields

$$a_{N+k}\leq a_N q^k.$$

The resulting geometric comparison establishes convergence.

If $L>1$, the terms eventually increase and therefore cannot tend to zero. The series consequently diverges.

A more general ratio test uses the upper limit of a_{n+1}/a_n, but the present comparison primarily considers the ordinary-limit formulation.

The Cauchy root test examines

$$L=\limsup_{n\to\infty}\sqrt[n]{a_n}.$$

Its conclusions are

$$\begin{cases} L<1 & \text{the series converges,} \\ L>1 & \text{the series diverges,} \\ L=1 & \text{the test is inconclusive.} \end{cases}$$

For $L<1$, one can select q with

$$L<q<1$$

such that eventually

$$a_n\leq q^n.$$

Thus, the series converges by comparison with a geometric series.

If $L>1$, infinitely many terms satisfy $\sqrt[n]{a_n}>1+\varepsilon$ for some $\varepsilon>0$, so the terms do not approach zero.

Unlike the ordinary ratio test, the root test can be applied effectively to certain sequences exhibiting oscillations in their consecutive-term ratios.

The Raabe test provides a refinement of the ratio test when

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1.$$

Define

$$R=\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right).$$

When this limit exists, the classical Raabe criterion states that

$$\begin{cases} R>1 & \text{the series converges,} \\ R<1 & \text{the series diverges,} \\ R=1 & \text{the test is inconclusive.} \end{cases}$$

The expression

$$n\left(\frac{a_n}{a_{n+1}}-1\right)$$

measures the first-order deviation of the consecutive-term ratio from unity.

This makes the test especially useful for terms that decay approximately as powers of n.

For instance, if

$$a_n=\frac{1}{n^p},$$

then

$$\frac{a_n}{a_{n+1}}=\left(1+\frac{1}{n}\right)^p.$$

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Using the expansion

$$\left(1+\frac{1}{n}\right)^{p}=1+\frac{p}{n}+O\left(\frac{1}{n^{2}}\right),$$

we obtain

$$R=p.$$

Therefore, the Raabe test reproduces the classical convergence threshold of the p-series whenever $p\neq 1$.

The Gauss test provides a more refined asymptotic criterion.

Suppose the positive terms satisfy

$$\frac{a_n}{a_{n+1}}=1+\frac{\alpha}{n}+O\left(\frac{1}{n^{1+\delta}}\right), \qquad \delta>0.$$

Under this assumption,

$$\begin{cases}\alpha>1 & \text{the series converges,}\\ \alpha\leq 1 & \text{the series diverges.}\end{cases}$$

An equivalent classical formulation assumes

$$\frac{a_n}{a_{n+1}}=1+\frac{\alpha}{n}+\frac{b_n}{n^{\beta}},$$

where $\beta>1$ and the sequence $\{b_n\}$ is bounded.

The essential distinction between the Gauss and Raabe tests occurs when the first-order coefficient equals one.

If

$$\alpha=1$$

and the remainder satisfies the required higher-order estimate, the Gauss test establishes divergence, whereas the Raabe test alone is inconclusive.

The remainder condition is essential. Without it, the conclusion at the critical value $\alpha=1$ is not generally valid.

Results. The mathematical comparison reveals significant differences in the information required and the classes of series for which each criterion is effective.

Criterion	Main expression	Convergence condition	Inconclusive case
d'Alembert	$\lim a_{n+1}/a_n=L$	$L<1$	$L=1$
Cauchy	$\limsup\sqrt[n]{a_n}=L$	$L<1$	$L=1$
Raabe	$\lim n(a_n/a_{n+1}-1)=R$	$R>1$	$R=1$
Gauss	$a_n/a_{n+1}=1+\alpha/n+O(r$	$\alpha>1$	No undecided value under the stated expansion

The results indicate that the four tests do not possess identical levels of applicability. In particular, the Cauchy root test is stronger than the ratio test in the following precise sense: whenever the consecutive-term ratio has a limit L, the n-th root also approaches L. However, the converse is not necessarily true.

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The Raabe test provides additional information in many cases where both exponential-type criteria are inconclusive.

The Gauss test can resolve the critical Raabe case, but only when its stronger asymptotic assumptions are satisfied.

Consider

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

Let

$$a_n = \frac{n!}{n^n}.$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n.$$

Consequently,

$$L = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e}.$$

Since

$$\frac{1}{e} < 1,$$

the series converges by the d'Alembert test.

The Cauchy test also establishes convergence because Stirling's formula gives

$$\sqrt[n]{\frac{n!}{n^n}} \to \frac{1}{e}.$$

This example demonstrates the effectiveness of the ratio and root tests for factorial-exponential expressions.

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p>0.$$

For the ratio test,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1.$$

Thus, the ratio test is inconclusive.

Similarly,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} n^{-p/n} = 1.$$

The Cauchy test is also inconclusive.

However, the Raabe test yields

$$R = \lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)^p - 1\right] = p.$$

Hence,

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$$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{converges}, & p>1,\\ \text{diverges}, & 0<p<1.\end{cases}$$

At $p=1$, the Raabe test is inconclusive.
The Gauss test resolves this case because

$$\frac{a_n}{a_{n+1}}=1+\frac{1}{n},$$

which satisfies its assumptions with $\alpha=1$ and zero remainder.
Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

diverges.

Convergence threshold for the *p*-series

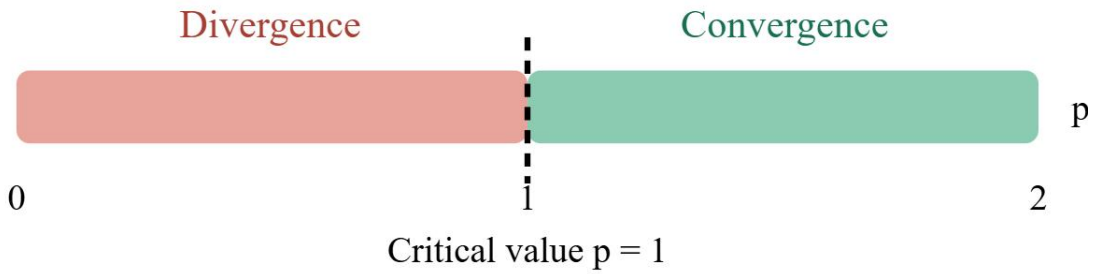


Figure 1. Convergence classification of the positive p-series. The boundary $p=1$ belongs to the divergent region.

Consider

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p},\qquad p\in\mathbb{R}.$$

Let

$$a_n=\frac{1}{n(\ln n)^p}.$$

The ratio of consecutive terms is

$$\frac{a_n}{a_{n+1}}=\left(1+\frac{1}{n}\right)\left(\frac{\ln(n+1)}{\ln n}\right)^p.$$

As $n\to\infty$,

$$\frac{a_n}{a_{n+1}}=1+\frac{1}{n}+\frac{p}{n\ln n}+O\left(\frac{1}{n^2}\right),$$

for each fixed p.
It follows that

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$$R=\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=1.$$

Therefore, the Raabe test is inconclusive for every value of p.

The d'Alembert and Cauchy tests are also inconclusive.

Moreover, when $p\neq 0$, the term

$$\frac{p}{n\ln n}$$

cannot be absorbed into an error term $O(n^{-1-\delta})$ for any $\delta>0$.

Consequently, the Gauss test in its stated form cannot resolve this example for $p\neq 0$.

Using the integral test,

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p},$$

and the substitution $t=\ln x$, we obtain

$$\int_{\ln 2}^{\infty} \frac{dt}{t^p}.$$

Thus,

$$\boxed{\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges if and only if } p>1.}$$

This example demonstrates that even the Gauss test does not resolve every borderline case.

Consider the positive sequence

$$a_n=\begin{cases}2^{-n}, & n \text{ even},\\ 3^{-n}, & n \text{ odd}.\end{cases}$$

The consecutive-term ratio does not have a finite limit. Indeed, along alternating subsequences, the ratios tend to zero or grow without bound.

Therefore, the ordinary d'Alembert ratio test cannot be applied conclusively.

However,

$$\sqrt[n]{a_n}=\begin{cases}\frac{1}{2}, & n \text{ even},\\ \frac{1}{3}, & n \text{ odd}.\end{cases}$$

Hence,

$$\limsup_{n\to\infty} \sqrt[n]{a_n}=\frac{1}{2}<1.$$

By the Cauchy root test, the series converges.

This example establishes that the root test can succeed even when the ordinary ratio limit fails to exist.

Series	d'Alembert	Cauchy	Raabe	Gauss
$\sum n!/n^n$	Converges	Converges	Not needed	Not needed
$\sum 1/n^2$	Inconclusive	Inconclusive	Converges	Converges

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Series	d'Alembert	Cauchy	Raabe	Gauss
$\sum 1/\sqrt{n}$	Inconclusive	Inconclusive	Diverges	Diverges
$\sum 1/n$	Inconclusive	Inconclusive	Inconclusive	Diverges
$\sum_{n\geq 2} 1/[n(\ln n)$	Inconclusive	Inconclusive	Inconclusive	Assumptions fail
Alternating exponential-rate example in Section 3.5	Ratio limit fails	Converges	Not applicable	Not applicable

The table illustrates that no single criterion is universally decisive for all positive-term series.

Discussion. The comparative analysis demonstrates that the effectiveness of a convergence criterion depends on the asymptotic structure of the general term rather than simply on the algebraic complexity of its expression.

The ratio and root tests are closely connected through the asymptotic behavior of positive sequences.

If

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=L, \qquad 0\leq L\leq\infty,$$

then, under the usual extended-limit interpretation,

$$\lim_{n\to\infty}\sqrt[n]{a_n}=L.$$

For finite positive L, this relationship can be understood by writing

$$a_n=a_1\prod_{k=1}^{n-1}\frac{a_{k+1}}{a_k}$$

and applying logarithms and Cesàro averaging.

Consequently, whenever the ordinary ratio test produces a decisive conclusion, the root test produces the same conclusion.

However, the reverse implication does not hold. The root test can remain effective for sequences whose consecutive-term ratios oscillate.

This mathematical relationship provides a precise explanation of the broader applicability of the Cauchy test.

The Raabe test examines asymptotic behavior at a finer scale.

If

$$\frac{a_n}{a_{n+1}}=1+\frac{\alpha}{n}+o\left(\frac{1}{n}\right),$$

then

$$R=\alpha.$$

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The parameter α describes the leading-order rate at which the terms decrease relative to their indices.

For $\alpha>1$, the decay is sufficiently rapid to ensure convergence.

For $\alpha<1$, the decay is insufficient, and divergence follows.

At $\alpha=1$, the leading-order information alone does not distinguish between convergent and divergent series.

For example, both

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$$

and

$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$$

have Raabe limit equal to one, although the first converges and the second diverges.

Therefore, the critical value represents a genuine limitation of first-order asymptotic analysis.

The Gauss test improves the analysis of the critical case by imposing stronger control on the error term.

Suppose

$$\frac{a_n}{a_{n+1}}=1+\frac{1}{n}+O(n^{-1-\delta}),\qquad \delta>0.$$

Taking logarithms gives

$$\ln\frac{a_n}{a_{n+1}}=\frac{1}{n}+O(n^{-1-\varepsilon})$$

for some $\varepsilon>0$.

Summing these relations yields

$$\ln a_n=-\ln n+C+o(1)$$

for an appropriate constant C.

Consequently,

$$a_n\sim\frac{K}{n},\qquad K>0.$$

The series therefore diverges by comparison with the harmonic series.

This argument explains why the Gauss test can establish divergence at the critical coefficient $\alpha=1$.

However, when the remainder contains logarithmic corrections of order $1/(n\ln n)$, the required summability condition fails. Such cases demand more refined tests, such as Bertrand's test or an appropriate direct comparison.

Conclusion. This study presented a systematic comparative analysis of the d'Alembert, Cauchy, Raabe, and Gauss tests for positive-term infinite series.

The mathematical investigation supports five principal conclusions. Overall, the study establishes that the effective analysis of positive-term series requires a systematic understanding of asymptotic behavior and careful verification of the assumptions underlying each convergence test. The results provide a theoretical basis for improving convergence-test selection and may be useful in

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university-level mathematical analysis, calculus instruction, and the development of problem-solving methodologies.

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