HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

# APPLICATION OF LINEAR OPERATORS AND TENSOR PRODUCTS IN QUANTUM ENTANGLEMENT AND QUANTUM COMPUTING

**Sevinchova Charosxon Maxmudjonovna**
New Uzbekistan University
E-mail: c.sevinchova@newuu.uz
Doi: https://doi.org/10.5281/zenodo.23229871

**Abstract:** Quantum computing is fundamentally based on the mathematical structure of complex Hilbert spaces, linear operators, tensor products, and unitary transformations. Among these concepts, tensor products provide the formal mechanism for constructing the state space of composite quantum systems, whereas linear and unitary operators describe quantum transformations, measurements, and computational gates. This article examines the mathematical relationship between linear operators and tensor products and analyzes their role in the formation, manipulation, and characterization of quantum entanglement. Particular attention is given to two-qubit systems, separable and entangled states, Schmidt decomposition, controlled operations, and the role of non-factorizable operators in quantum circuits. The analysis demonstrates that tensor products perform a dual function: they provide the mathematical representation of composite systems and enable the construction of local operations acting on individual subsystems. At the same time, genuinely entangling transformations, such as the controlled-NOT operation, cannot in general be represented as a single tensor product of independent local operators. Recent research further demonstrates that tensor-product and operator-Schmidt decompositions can be used to characterize operator entanglement, reduce circuit complexity, and construct efficient representations of quantum processes. The results establish that linear algebra and tensor-product structures are not merely formal tools but constitute the mathematical foundation connecting quantum entanglement with quantum computation.

**Keywords:** quantum computing, quantum entanglement, linear operators, tensor product, Hilbert space, unitary operator, quantum gates, Schmidt decomposition, CNOT, tensor networks.

## 1. INTRODUCTION

Quantum computing represents a computational paradigm in which information is encoded, transformed, and measured according to the principles of quantum mechanics. Unlike classical bits, whose states are represented by the discrete values 0 and 1, a qubit is described by a vector in a complex Hilbert space. The mathematical structure of multiple-qubit systems therefore requires a formalism capable of representing both independent subsystems and correlations that have no classical counterpart.

The tensor product provides precisely this structure. If the states of two quantum systems are represented in Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the composite system is represented in

$$\mathcal{H}_{AB}=\mathcal{H}_A \otimes \mathcal{H}_B .$$

This construction is fundamental because the dimension of the composite space is the product of the dimensions of the constituent spaces. For two qubits,

$$\mathcal{H}_{AB}=\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4,$$

with computational basis

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

$$\{ | 00\rangle, | 01\rangle, | 10\rangle, | 11\rangle \}.$$

The tensor-product formalism therefore allows quantum information to be represented in a mathematically consistent way when several qubits interact. Modern quantum-information treatments explicitly define the composite Hilbert space through tensor products and show how operators on individual subsystems are lifted to operators on the entire composite system.

However, tensor products have a deeper significance. Not every vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be expressed as

$$| \psi_A\rangle \otimes | \psi_B\rangle.$$

States that cannot be represented in this form are entangled. Consequently, the same mathematical structure that constructs the state space of a composite quantum system also creates the possibility of quantum entanglement.

The objective of this article is to analyze how linear operators and tensor products jointly provide the mathematical framework for quantum entanglement and quantum computation, with particular emphasis on the relationship between local operations, nonlocal transformations, and entangling quantum gates.

## 2. MATERIALS AND METHODS

The study employs a theoretical and mathematical analysis of the operator formalism of quantum mechanics and quantum information theory. The methodological basis consists of linear algebra, Hilbert-space theory, tensor-product algebra, operator decomposition, and quantum-circuit analysis.

The analysis proceeds through four interconnected stages. First, the mathematical representation of individual and composite quantum systems is established. Second, tensor products of vectors and linear operators are examined. Third, separable and entangled states are distinguished through factorization and Schmidt decomposition. Finally, the role of unitary and controlled operators in quantum computation is analyzed.

The mathematical framework is complemented by recent literature on tensor-product decomposition, tensor networks, entanglement characterization, and quantum circuit construction. Recent research demonstrates that operator-Schmidt decomposition can be used to investigate the entangling properties of quantum operations and to develop low-rank approximations and lower-depth implementations of quantum circuit unitaries.

## 3. RESULTS

### 3.1. Tensor products as the mathematical structure of composite quantum systems

Let

$$| \psi_A\rangle \in \mathcal{H}_A, \quad | \psi_B\rangle \in \mathcal{H}_B.$$

The state of the combined system is represented by

$$| \psi_A\rangle \otimes | \psi_B\rangle.$$

For two qubits,

$$| \psi_A\rangle = \alpha | 0\rangle + \beta | 1\rangle,$$

and

$$| \psi_B\rangle = \gamma | 0\rangle + \delta | 1\rangle.$$

Their tensor product is

$$| \psi_A\rangle \otimes | \psi_B\rangle = \alpha\gamma | 00\rangle + \alpha\delta | 01\rangle + \beta\gamma | 10\rangle + \beta\delta | 11\rangle.$$

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

Thus, the tensor product does not merely concatenate two states; it constructs a higher-dimensional state space in which correlations between the subsystems can be represented.

The corresponding operator construction is equally important. Given

$$A:\mathcal{H}_A\rightarrow\mathcal{H}_A$$

and

$$B:\mathcal{H}_B\rightarrow\mathcal{H}_B,$$

their tensor product

$$A\otimes B$$

acts according to

$$(A\otimes B)(\mid\psi_A\rangle\otimes\mid\psi_B\rangle)=(A\mid\psi_A\rangle)\otimes(B\mid\psi_B\rangle).$$

This property enables quantum circuits to apply different operations to different subsystems simultaneously. The mathematical definition and matrix representation of such operators are standard components of quantum-information theory.

For matrices $A$and $B$, the Kronecker-product representation can be written as

$$A\otimes B=\begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Therefore, tensor multiplication converts local operators into operators acting on the joint Hilbert space.

**3.2. Linear operators and unitary transformations**

Quantum computation is fundamentally a sequence of transformations of quantum states. In a closed quantum system, these transformations are represented by unitary operators satisfying

$$U^{\dagger}U=UU^{\dagger}=I.$$

If the initial state is $\mid\psi\rangle$, its evolution under a quantum gate $U$is

$$\mid\psi'\rangle=U\mid\psi\rangle.$$

The unitary condition preserves normalization:

$$\langle\psi'\mid\psi'\rangle=\langle\psi\mid U^{\dagger}U\mid\psi\rangle=\langle\psi\mid\psi\rangle.$$

For multiple qubits, local transformations can be represented as

$$U_A\otimes U_B.$$

Such an operator transforms a product state into another product state:

$$(U_A\otimes U_B)(\mid\psi_A\rangle\otimes\mid\psi_B\rangle)=U_A\mid\psi_A\rangle\otimes U_B\mid\psi_B\rangle.$$

This observation is central to understanding why genuinely entangling operations require a more general class of operators.

**3.3. Separable and entangled states**

A two-qubit state

$$\mid\psi\rangle=a\mid 00\rangle+b\mid 01\rangle+c\mid 10\rangle+d\mid 11\rangle$$

is separable if there exist single-qubit states

$$\mid\phi_A\rangle=\alpha\mid 0\rangle+\beta\mid 1\rangle$$

and

$$\mid\phi_B\rangle=\gamma\mid 0\rangle+\delta\mid 1\rangle$$

such that

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

$$|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle.$$

For the coefficient matrix

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

a two-qubit pure state is separable precisely when

$$\det C = ad - bc = 0.$$

If

$$ad - bc \neq 0,$$

the state cannot be factorized into two independent single-qubit states and is therefore entangled.

A canonical example is the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

There are no single-qubit states $|\phi_A\rangle$ and $|\phi_B\rangle$ satisfying

$$|\Phi^+\rangle = |\phi_A\rangle \otimes |\phi_B\rangle.$$

This inability to factorize is the mathematical signature of entanglement.

**3.4. Schmidt decomposition and the structure of entanglement**

For a bipartite pure state, the Schmidt decomposition provides a particularly useful representation:

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |u_i\rangle \otimes |v_i\rangle,$$

where

$$\lambda_i \geq 0, \quad \sum_i \lambda_i = 1.$$

The integer $r$, known as the Schmidt rank, provides a direct criterion for entanglement.

If

$$r = 1,$$

the state is separable.

If

$$r > 1,$$

the state is entangled.

For the Bell state,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} |0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |1\rangle,$$

and therefore its Schmidt rank equals 2.

The Schmidt decomposition is not only a theoretical characterization of state entanglement. It also forms the conceptual basis for operator-level analyses of entangling quantum transformations. Recent PRX Quantum research explicitly connects state Schmidt decomposition with operator-Schmidt or tensor-product decomposition for analyzing the entangling characteristics of quantum operations.

**3.5. Linear operators as generators of quantum entanglement**

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

An important distinction must be made between local tensor-product operators and genuinely entangling operators.

Suppose

$$U=U_A \otimes U_B .$$

For every product state,

$$|\psi_A\rangle \otimes |\psi_B\rangle,$$

the result remains a product state:

$$U_A |\psi_A\rangle \otimes U_B |\psi_B\rangle.$$

Consequently, a single operator of this form cannot transform a separable state into an entangled state.

Entanglement generation therefore requires an operator that cannot, in general, be expressed as a single tensor product of independent local operators.

The controlled-NOT operator provides the standard example:

$$CNOT|00\rangle=|00\rangle,$$
$$CNOT|01\rangle=|01\rangle,$$
$$CNOT|10\rangle=|11\rangle,$$
$$CNOT|11\rangle=|10\rangle.$$

Its matrix is

$$CNOT=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

The importance of CNOT becomes clear when it acts on

$$|+\rangle \otimes |0\rangle,$$

where

$$|+\rangle=\frac{|0\rangle+|1\rangle}{\sqrt{2}}.$$

We obtain

$$CNOT(|+\rangle \otimes |0\rangle)=\frac{|00\rangle+|11\rangle}{\sqrt{2}}.$$

The output is a Bell state and is therefore entangled.

Thus, a simple sequence of a one-qubit superposition operation followed by a two-qubit entangling operator creates a nonseparable quantum state.

**3.6. Tensor products in quantum circuits**

For an $n$-qubit register,

$$\mathcal{H}=\bigotimes_{k=1}^{n}\mathbb{C}^2 .$$

Its dimension is

$$\dim_{f_0}(\mathcal{H})=2^n.$$

This exponential growth is one of the defining mathematical features of quantum information. A general state can be written as

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

$$|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle,$$

with

$$\sum_x |\alpha_x|^2 = 1.$$

Quantum gates act on this exponentially large state space. A gate acting only on qubit $k$ is embedded into the complete system through tensor products:

$$U_k = I \otimes \cdots \otimes I \otimes U \otimes I \otimes \cdots \otimes I.$$

This formalism allows local transformations to be represented consistently within the global state space.

The tensor-product construction therefore links the microscopic action of individual gates with the global evolution of a quantum register.

**3.7. Operator-Schmidt decomposition**

The same mathematical idea can be extended from quantum states to quantum operators. A bipartite operator can, under appropriate conditions, be represented as

$$U = \sum_{k=1}^{R} s_k A_k \otimes B_k,$$

where $A_k$ and $B_k$ are operators on the respective subsystems.

The smallest number $R$ of terms required in such a representation is related to the operator-Schmidt rank.

This concept is particularly important because a quantum operation may not be a single tensor product even though it can be represented as a sum of tensor products.

Recent research has demonstrated that tensor-product decomposition of quantum operations can be used to analyze operator nonlocality, construct low-rank approximations, and investigate lower-depth implementations of quantum-circuit unitaries.

This creates an important bridge between abstract linear algebra and practical quantum-computing optimization.

**3.8. Tensor networks and scalable quantum computation**

The tensor-product structure naturally leads to tensor-network representations. Instead of explicitly storing the full $2^n$-dimensional state vector, certain quantum states and operators can be represented through networks of smaller tensors.

Tensor networks have become important tools for quantum-computation simulation, quantum-circuit synthesis, quantum error correction and quantum machine learning. A 2025 review in *Nature Reviews Physics* emphasizes that tensor-network methods provide structured and compressed representations of quantum states and processes and surveys their applications across several areas of quantum computing [1, pp. 581–593].

The relevance of this approach becomes especially clear for many-body systems. If a quantum state possesses limited entanglement across appropriate partitions, tensor-network representations may dramatically reduce the effective computational resources required for its classical simulation.

Consequently, entanglement is simultaneously:

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

1. a resource for quantum computation;
2. a source of complexity for classical simulation;
3. a structural quantity exploited by tensor-network methods.

This apparent duality is one of the most important conceptual connections between quantum information theory and computational mathematics.

**4. DISCUSSION**

The analysis demonstrates that tensor products and linear operators should not be considered independent mathematical tools in quantum computing. Their roles are complementary.

The tensor product establishes the architecture of composite quantum systems. Without it, there would be no mathematically consistent representation of multi-qubit states. Linear operators, particularly unitary operators, determine how these states evolve. Entangling operators then create correlations that cannot be described through independent local states.

The distinction between

$$U_A \otimes U_B$$

and a genuinely entangling operator such as CNOT is therefore fundamental. The former describes independent local transformations, whereas the latter can generate correlations that change the structure of the global state.

This distinction also clarifies an important feature of quantum advantage. The computational power of a quantum circuit does not arise simply from increasing the number of states in a mathematical vector. Rather, useful quantum algorithms exploit superposition, interference, entanglement, and carefully structured unitary transformations.

Recent work on quantum tensor-product decomposition further indicates that operator-level entanglement is relevant to understanding the complexity of quantum transformations. At the same time, tensor-network research shows that the structure of entanglement can be exploited to represent and simulate quantum processes more efficiently.

Another important implication concerns quantum-circuit optimization. If a complicated operator can be approximated or decomposed into a relatively small number of tensor-product components, its implementation may require fewer resources. This is particularly important for near-term quantum computers, where circuit depth, gate count, connectivity, and noise impose strong practical constraints.

Recent research on hardware-efficient quantum circuits demonstrates the continuing importance of minimizing gate resources and adapting circuit structures to the physical characteristics of quantum processors.

Thus, the mathematical study of tensor products and linear operators is directly connected to contemporary problems in quantum hardware and algorithm design.

**5. CONCLUSION**

Linear operators and tensor products constitute two fundamental mathematical pillars of quantum computing. Tensor products provide the state-space construction for composite quantum systems, while linear and unitary operators describe the transformations performed on these systems.

The analysis shows that a product state can be written as

$$|\psi_A\rangle \otimes |\psi_B\rangle,$$

HSR (LONDON),
HOUGHTON STREET REVIEW JOURNAL
ISSN: 2058-5519|Impact factor-15.9
http://houghtonstreetreview.com/index.php/hsr

whereas an entangled state cannot generally be reduced to such a single tensor product. Schmidt decomposition provides a rigorous method for identifying this distinction, while operator-Schmidt decomposition extends the same conceptual framework to quantum transformations.

The CNOT gate demonstrates the computational significance of these principles. By acting on a superposition state, it can generate a Bell state, thereby converting local quantum information into a genuinely entangled global state. This demonstrates that entangling operators are indispensable components of quantum computation.

At a larger scale, tensor products generate the exponentially growing state spaces of multi-qubit registers, while tensor-network methods provide structured approaches for representing and manipulating some of these high-dimensional states and operators. Contemporary research confirms the relevance of these structures to quantum simulation, circuit synthesis, error correction, and quantum machine learning.

Therefore, the mathematical relationship between linear operators and tensor products is not merely foundational theory. It provides a unified framework for understanding how quantum information is represented, transformed, entangled, and computationally processed. Further research into operator entanglement, tensor-product decomposition, tensor networks, and resource-efficient quantum circuits may contribute to the development of more scalable quantum algorithms and hardware.

## REFERENCES

1. Berezutskii, A., Liu, M., Acharya, A., et al. (2025). Tensor networks for quantum computing. Nature Reviews Physics, 7, 581–593. https://doi.org/10.1038/s42254-025-00853-1.
2. Mansuroglu, R., et al. (2024). Quantum Tensor-Product Decomposition from Choi-State Tomography. PRX Quantum, 5, 030306. https://doi.org/10.1103/PRXQuantum.5.030306.
3. TU Delft. Mathematics for Quantum Physics: Tensor Products. Mathematical treatment of tensor-product Hilbert spaces and operators.
4. Lin, L. Quantum Algorithms for Scientific Computation. University of California, Berkeley lecture notes. See discussion of tensor products of linear operators and multi-qubit state spaces.
5. Semidefinite programming relaxations for quantum correlations. (2024). Reviews of Modern Physics, 96, 045006. https://doi.org/10.1103/RevModPhys.96.045006.
6. Quantification of entanglement and coherence with purity detection. (2024). npj Quantum Information, 10, 60.
7. Creating and controlling global Greenberger-Horne-Zeilinger entanglement on quantum processors. (2024). Nature Communications.
8. Approximate encoding of quantum states using shallow circuits. (2024). npj Quantum Information.
9. Efficient fault-tolerant implementations of non-Clifford gates with reconfigurable atom arrays. (2024). npj Quantum Information.
10. Tensor Network Enhanced Dynamic Multiproduct Formulas. (2025). PRX Quantum, 6, 020360. https://doi.org/10.1103/8bzc-dlgt.