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INVESTIGATION OF THE STABILITY OF INVERSE PROBLEMS FOR NONLINEAR EVOLUTION EQUATIONS

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Abstract. Inverse problems for nonlinear evolution equations play an important role in mathematical modeling, computational science, physics, engineering, environmental monitoring, and biomedical research. Unlike direct problems, in which the evolution of a system is determined from known initial and boundary data, inverse problems aim to reconstruct unknown parameters, sources, initial states, or boundary conditions from incomplete and noisy observations. A fundamental difficulty of such problems is instability: small perturbations in the measured data may cause significant changes in the reconstructed solution. This paper provides an analytical investigation of the stability of inverse problems associated with nonlinear evolution equations. The main sources of instability are examined, including nonlinearity, incomplete observations, measurement noise, ill-posedness, and sensitivity to initial approximations. Different approaches to stabilization are analyzed, with particular attention to regularization methods, iterative algorithms, variational techniques, and parameter constraints. The analytical comparison demonstrates that stability depends not only on the mathematical structure of the evolution equation but also on the observation operator, data quality, regularization strategy, and computational procedure.

Keywords: inverse problem, nonlinear evolution equation, stability, ill-posed problem, regularization, iterative method, noisy data, parameter identification.

Introduction. Inverse problems constitute an important class of mathematical problems in which unknown characteristics of a dynamical system are reconstructed from observed effects. They arise in many practical applications, including heat transfer, fluid dynamics, diffusion processes, population models, environmental processes, medical imaging, geophysical exploration, and material identification.

In a direct problem, the mathematical model and its parameters are assumed to be known, and the objective is to determine the evolution of the system. In an inverse problem, the situation is essentially reversed. The available information usually consists of measurements of the system at one or several moments, while some model parameters, source terms, initial conditions, or boundary characteristics remain unknown. The difficulty becomes substantially greater when the governing equation is nonlinear. Nonlinearity creates interactions between unknown quantities and the state of the system. As a consequence, a small error in observations may be amplified during the reconstruction process. This phenomenon is generally referred to as instability.

The concept of stability is particularly important in inverse problems because real measurements are never perfectly accurate. Experimental observations are affected by instrumental errors, environmental disturbances, discretization errors, numerical approximations, and incomplete information. Therefore, a theoretically correct reconstruction method may become practically useless if it is highly sensitive to small perturbations in the input data.

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The central objective of this paper is to analytically investigate the stability of inverse problems for nonlinear evolution equations. The study focuses on the principal factors that determine stability and compares different approaches used to reduce instability. Particular attention is given to the relationship between data perturbations, model nonlinearity, regularization, and iterative reconstruction. The exact form of this estimate depends on the problem, but its interpretation is general: the reconstruction should not amplify observational errors excessively.

For nonlinear evolution equations, however, such a simple stability relationship is often unavailable. The stability constant may become very large, may depend strongly on the solution, or may fail to exist globally. Therefore, one of the principal objectives of inverse problem analysis is to determine under what conditions stable reconstruction is possible.

Nonlinearity of the evolution equation. Nonlinearity is another fundamental source of instability. In linear problems, the relationship between unknown parameters and observations is comparatively straightforward. In nonlinear problems, changes in parameters may alter the state of the system in a complicated and non-proportional manner.

For example, a small change in a diffusion coefficient may produce a moderate change in the solution in one region of the parameter space but a substantial change in another region. Similarly, nonlinear reaction terms may create rapid transitions, saturation effects, or multiple possible states. This means that stability may be local rather than global.

Incomplete observations. In many practical problems, the entire solution (u(x,t)) cannot be measured. Instead, observations may be available only:

- at selected spatial points;
- at specific time moments;
- on part of the boundary;
- in an averaged form;
- through indirect physical measurements.

Such incomplete information can significantly reduce the identifiability of unknown parameters. If two different parameter values produce nearly identical observations, then distinguishing between them becomes difficult. In this case, the inverse problem may be unstable even when the forward evolution equation is well posed.

Sensitivity to initial approximations. Iterative methods require an initial approximation of the unknown parameter or solution. For nonlinear inverse problems, the quality of this initial approximation can strongly influence the reconstruction.

A sufficiently accurate initial approximation may lead to rapid convergence. However, a poor initial approximation can result in:

- slow convergence;
- convergence to an undesirable solution;
- oscillatory behavior;
- divergence;
- excessive sensitivity to noise.

Therefore, stability analysis should consider not only the observational data but also the behavior of the reconstruction algorithm with respect to initial conditions. The first approach is to examine how the solution depends on the observational data and unknown parameters. If small perturbations in the data produce small perturbations in the solution, the problem possesses a form of continuous

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dependence. For nonlinear equations, this analysis is often local. One investigates the behavior of the solution near a particular parameter value or reference trajectory. This approach is theoretically valuable because it provides information about whether the inverse problem is fundamentally stable before numerical algorithms are applied.

Linearization. A common strategy for nonlinear problems is to analyze a linearized version of the model around a reference solution.

Suppose (u^) and (\lambda^) represent an approximate solution and parameter. Small perturbations can be introduced around this state. The nonlinear problem is then approximated by a linear sensitivity problem.

The resulting linearized model describes how small changes in the unknown parameter influence the observations.

This approach allows researchers to determine:

- which parameters are strongly identifiable;
- which parameters are weakly identifiable;
- whether the inverse problem is locally stable;
- which directions in parameter space are particularly sensitive to noise.

However, linearization is only a local approximation. Strong nonlinear effects may make conclusions based on linearization unreliable far from the reference solution. When the inverse problem is unstable, regularization becomes necessary.

The main principle of regularization is to replace the original unstable reconstruction problem with a modified problem that incorporates additional information about the expected solution.

Gradient-based methods. Gradient-based methods update the unknown parameter in the direction that reduces the discrepancy between predicted and observed data.

Their main advantages include:

- relatively simple implementation;
- moderate computational requirements;
- applicability to large-scale problems;
- compatibility with regularization.

However, their convergence may be slow, particularly when the inverse problem is poorly conditioned. From the stability perspective, gradient-based methods can be robust when the step size and stopping criterion are appropriately selected. Conjugate gradient approaches can accelerate convergence in many optimization problems. Compared with basic gradient descent, they use information from previous iterations to select more efficient search directions.

Their advantage becomes particularly noticeable when the inverse problem contains a large number of unknown parameters.

Nevertheless, acceleration may increase sensitivity to noisy data if regularization and stopping rules are not properly controlled.

Thus, computational speed and stability should be considered simultaneously rather than independently.

Gauss–Newton methods. Gauss–Newton methods are especially useful for nonlinear parameter identification problems. They rely on the local linearization of the nonlinear observation operator. When the initial approximation is sufficiently close to the true solution, convergence can be very fast.

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However, the same property creates a weakness: the method can be sensitive to the initial approximation.

If the problem is strongly nonlinear or poorly conditioned, the linearized system may provide an inaccurate representation of the actual problem. Therefore, Gauss–Newton methods are generally more effective when the nonlinear model is reasonably well behaved and a good initial estimate is available.

Levenberg–Marquardt methods. Levenberg–Marquardt methods combine characteristics of gradient-type and Gauss–Newton approaches.

When the current approximation is far from the solution, the method behaves more conservatively. As the approximation improves, it can gradually adopt a faster Gauss–Newton-type behavior. This adaptive character makes the method attractive for nonlinear inverse problems. From the stability perspective, its main advantage is the ability to reduce the influence of poorly conditioned directions during early iterations.

The main algorithms can be compared according to several criteria.

Method	Convergence Speed	Stability	Dependence on Initial Guess	Computational Cost
Gradient method	Low–moderate	High	Low–moderate	Low
Conjugate gradient	High	Moderate	Moderate	Moderate
Gauss–Newton	High	Moderate	High	High
Levenberg–Marquardt	Moderate–high	High	Moderate	High
Regularized iterative method	Moderate	Very high	Moderate	Moderate–high

This comparison shows that no method simultaneously provides maximum convergence speed, maximum stability, minimum computational cost, and complete resistance to noise. For example, gradient methods are generally more conservative and stable, but they may require many iterations. Gauss–Newton methods can be significantly faster, but they may become unstable when the initial approximation is poor or the problem is strongly nonlinear.

Levenberg–Marquardt methods often provide a compromise between these two extremes. Regularized iterative approaches are particularly valuable when the observational data contain significant noise.

Influence of regularization on stability. Regularization is not merely a numerical technique; it can be interpreted as a mechanism for incorporating additional information into an ill-posed inverse problem.

Suppose the observational data are highly noisy. An algorithm that attempts to reproduce every fluctuation in the data may reconstruct the noise rather than the actual physical parameter.

This phenomenon is known as overfitting.

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Regularization prevents excessive adaptation to insignificant fluctuations by imposing additional restrictions on the reconstructed solution.

Typical regularization principles include:

- smoothness;
- boundedness;
- sparsity;
- physical constraints;
- prior information;
- temporal consistency;
- spatial consistency.

The choice of regularization should reflect the physical characteristics of the unknown parameter. For example, if the parameter is expected to vary smoothly in space, a smoothness-based regularization is appropriate. If the parameter represents a localized source, a sparsity-promoting approach may be more suitable.

Effect of the observation time on stability. The time at which measurements are collected can have a significant influence on the stability of the inverse problem.

Early-time observations may contain strong information about initial conditions or rapidly changing parameters. However, they may also be highly sensitive to measurement errors. At later times, diffusion and dissipative processes may smooth the solution. This can improve numerical behavior but simultaneously reduce information about fine-scale characteristics of the original state.

The optimal observation strategy should consequently consider not only the amount of data but also the timing and location of measurements. This observation leads to an important conclusion: improving the stability of an inverse problem does not necessarily require a more sophisticated numerical algorithm. In some situations, better experimental design can provide a greater improvement.

For weakly nonlinear systems, linearization may provide an adequate approximation. In this case, classical optimization and regularization techniques can often be applied successfully.

For strongly nonlinear systems, however, several difficulties may arise:

1. multiple local minima;
2. strong dependence on initial conditions;
3. non-unique parameter reconstructions;
4. rapidly changing sensitivities;
5. possible bifurcation effects;
6. increased computational complexity.

As a result, stability should be interpreted locally and in relation to the specific region of the parameter space.

Analytical discussion. The analysis demonstrates that stability in inverse problems for nonlinear evolution equations is determined by several interconnected factors.

First, the mathematical structure of the forward model determines how strongly the unknown parameters influence the observations. If the observation operator is weakly sensitive to a particular parameter, accurate reconstruction may be impossible even with a sophisticated algorithm.

Second, the quality of the observational data plays a decisive role. High noise levels inevitably reduce the amount of reliable information available for reconstruction.

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Third, the algorithm influences how errors are propagated. Aggressive optimization methods may achieve rapid convergence but can also amplify noise. Conservative methods may provide greater stability but require more computational effort.

Fourth, regularization introduces additional assumptions that can transform an unstable inverse problem into a practically solvable one. However, excessive regularization can suppress genuine features of the solution. A reconstruction that reproduces noisy measurements perfectly may be less physically meaningful than a smoother reconstruction that slightly deviates from the observations. Therefore, the quality of an inverse solution should be evaluated using multiple criteria rather than a single residual measure.

Practical recommendations. Based on the analytical investigation, several recommendations can be formulated.

Recommendation 1: Analyze stability before numerical reconstruction. Before implementing an iterative algorithm, the sensitivity of the inverse problem to observational perturbations should be investigated.

Recommendation 2: Estimate the noise level. The expected measurement error should be incorporated into the reconstruction strategy.

Recommendation 3: Use regularization for ill-posed problems. If small perturbations in the data produce large parameter changes, regularization should be introduced.

Recommendation 4: Select the algorithm according to the problem structure. Gradient methods are appropriate when stability and simplicity are priorities. Gauss–Newton methods are advantageous when a good initial approximation is available. Levenberg–Marquardt methods are useful when a compromise between stability and speed is required.

Recommendation 5: Avoid excessive iteration. For noisy data, unlimited iteration may lead to overfitting. A suitable stopping criterion should therefore be established.

Recommendation 6: Improve the observation design. Whenever possible, measurement locations and observation times should be selected to maximize information about the unknown parameters.

Recommendation 7: Incorporate physical constraints. Known physical properties such as positivity, boundedness, smoothness, or conservation laws can significantly improve reconstruction stability.

Results. The analytical comparison leads to several main results.

First, inverse problems for nonlinear evolution equations are generally more sensitive to perturbations than their corresponding direct problems.

Second, instability is caused not by nonlinearity alone but by the interaction between nonlinearity, incomplete observations, measurement noise, and the structure of the inverse mapping.

Third, iterative algorithms differ considerably in their stability characteristics. Simple gradient methods generally provide robust behavior but may converge slowly. Faster methods, such as Gauss–Newton, may provide efficient reconstruction but require better initial approximations and stronger control of instability.

Fourth, regularization substantially improves the reliability of reconstruction when the observational data are noisy.

Fifth, the stopping criterion has a direct influence on stability. Excessive iterations can transform an initially stable reconstruction into an unstable one by fitting measurement noise.

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Finally, stability should be regarded as a combined property of the mathematical model, observation process, reconstruction algorithm, regularization strategy, and computational implementation.

Regularization remains one of the most effective tools for stabilizing ill-posed inverse problems. Nevertheless, the choice of regularization must be consistent with the physical and mathematical characteristics of the unknown quantity.

The overall analysis indicates that there is no universally optimal reconstruction algorithm. The most appropriate method should be selected according to the degree of nonlinearity, noise level, amount of observational information, initial approximation, and computational resources. Future research may focus on adaptive regularization, data-driven parameter selection, hybrid iterative methods, uncertainty quantification, and machine-learning-assisted reconstruction for nonlinear evolution equations. Such approaches may provide more reliable and computationally efficient solutions to complex inverse problems arising in modern scientific and engineering applications.

Conclusion. This paper has presented an analytical investigation of the stability of inverse problems for nonlinear evolution equations. The analysis confirms that stability is one of the principal challenges in nonlinear parameter reconstruction and cannot be treated independently of the quality of observational data and the structure of the mathematical model.

The main sources of instability include measurement noise, incomplete observations, strong nonlinearities, ill-conditioning, and sensitivity to initial approximations. Different iterative approaches respond to these difficulties in different ways. Gradient-based methods are relatively stable and computationally simple, while Gauss–Newton methods provide faster local convergence at the cost of greater sensitivity. Levenberg–Marquardt methods offer a useful compromise between robustness and convergence speed.

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