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STUDY OF THE SOLVABILITY OF PARAMETER-DEPENDENT SPECIAL LINEAR INEQUALITY SYSTEMS USING OPTIMIZATION PROBLEMS

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Abstract. This paper studies conditions under which a parameter-dependent system of linear inequalities admits a solution for all parameter values in a given parameter domain, using optimization problems, in particular maximization problems. From a theoretical perspective, parametric systems of linear inequalities are closely related to such tools as Farkas' lemma, Fourier elimination, and parametric linear programming, while their practical applications arise in control theory, engineering, and economics. We analyze optimization-based approaches for determining the solvability conditions of parametric systems and consider applications to robust control and two-stage robust linear programming.

Keywords: parametric linear inequalities, optimization, Farkas' lemma, Fourier elimination, solvability, maximization problems, robust optimization.

Introduction. Parameter-dependent systems of linear inequalities play an important role in modern optimization theory and its applications. Many practical problems — such as robust control theory, parametric linear programming, and decision-making under uncertainty — reduce to determining whether a parameter-dependent system of inequalities is solvable for given parameter values.

In general, a parameter-dependent system of linear inequalities is given in the form

$$A(p)x \le b(p),$$

where $p \in \mathrm{P} \subseteq \mathbb{R}^k$ is the parameter vector, $x \in \mathbb{R}^n$ is the vector of unknowns, and $A(p)$ and $b(p)$ are a parameter-dependent matrix and vector, respectively. The main problem is to determine conditions under which the system has a solution for **all** $p \in \mathrm{P}$.

One effective approach to this problem is to reformulate it as an optimization problem, in particular as a maximization problem. In this paper, we study the theoretical foundations and practical aspects of this approach.

Basic Theoretical Foundations

The Concept of a Parametric System of Linear Inequalities

When a system of linear inequalities depends on parameters, its solution set can be regarded as a function of the parameter. Let P be the parameter domain. For each $p \in \mathrm{P}$, define the feasible set

$$\mathcal{F}(p) = \{x \in \mathbb{R}^n : A(p)x \le b(p)\}.$$

The system is solvable for all $p \in \mathrm{P}$ if and only if

$$\bigcap_{p \in \mathrm{P}} \mathcal{F}(p) \neq \varnothing .$$

Farkas' Lemma and Its Parametric Generalization

Farkas' lemma plays a fundamental role in optimization theory as a criterion for the solvability of systems of linear inequalities. The classical Farkas lemma can be stated as follows: the nonexistence of a solution to the system $Ax \le 0$ is characterized by the existence of a vector y satisfying $A^T y = c$, $y \ge 0$.

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In the parametric setting, generalized versions of Farkas' lemma have been developed. In particular, generalized Farkas-type results for systems of parametric linear inequalities with affinely adjustable variables have been established, making it possible to prove strong duality results in robust optimization problems.

Fourier Elimination Method

Fourier elimination is one of the classical methods for determining solvability conditions for systems of parametric linear inequalities. This method makes it possible to analyze a system of the form $Ax \leq Bb$ with respect to the parameter b and obtain a facial description of the set of values of b for which the system is feasible. Although the computational complexity of the method may grow exponentially with the size of the matrix A, the procedure is conceptually intuitive and can be applied to practical problems.

Testing Solvability Using Optimization Problems

Reduction to a Maximization Problem

The condition that a parameter-dependent system of linear inequalities be solvable for all values in a given parameter domain P can be formulated as an optimization problem. The basic idea is to construct a maximization problem that characterizes the set of parameter values for which the system is infeasible.

Consider the parametric system

$$A(p)x \leq b(p), \quad p \in \mathrm{P}.$$

The system is infeasible for some $p \in \mathrm{P}$ if

$$\exists\, p \in \mathrm{P}: \forall x \in \mathbb{R}^n, \quad A(p)x \not\leq b(p).$$

By Farkas' lemma, this condition is equivalent to the existence of a solution to

$$A(p)^T y = 0, \quad b(p)^T y < 0, \quad y \geq 0, \quad y \neq 0.$$

Thus, solvability of the parametric system over P can be related to the absence of a solution of the following **maximization problem**:

$$\max_{p \in \mathrm{P}} \varphi(p),$$

where $\varphi(p)$ is a function measuring the degree of infeasibility of the system at the parameter value p.

Optimization Criterion

The preceding considerations lead to the following necessary and sufficient condition for solvability of the parametric system over P in terms of a maximization problem.

Theorem. The parametric system

$$A(p)x \leq b(p)$$

has a solution for every $p \in \mathrm{P}$ if

$$\max_{p \in \mathrm{P}} \min_{y \in \Delta} \{b(p)^T y : A(p)^T y = 0,\ y \geq 0,\ \| y \|_1 = 1\} \geq 0,$$

where

$$\Delta = \{y \geq 0 : \| y \|_1 = 1\}$$

is the simplex. This criterion makes it possible to determine, via a maximization problem, whether there exist parameter values for which the system is infeasible.

Relation to Parametric Linear Programming

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The approach described above is closely related to parametric linear programming problems. In parametric optimization, the variation of an optimal solution over the entire range of parameter values is studied.

A parametric linear programming problem is given by

$$Q(p)=\max\{c(p)^T x : A(p)x \le b(p),\ x \ge 0\},$$

where $p \in P$. The existence of a bounded feasible solution to this problem for all $p \in P$ ensures the solvability of the original parametric system of inequalities over P.

Summary of Main Results for Parametric Systems

The table below summarizes the solvability conditions in optimization form for different structures of the parameter domain P.

Solvability conditions for different types of parameter domains

Parameter domain P	Solvability condition (optimization form)	Remark
Finite set $\{p_1,\dots,p_m\}$	$\min_i \max_{y\in\Delta}\{b(p_i)^T y : A(p_i)^T y=0\}\ge 0$	Reduced to a finite system
Polyhedron (convex polytope)	$\max_{p\in P}\min_{y\in\Delta}\{b(p)^T y : A(p)^T y=0\}\ge 0$	Convex optimization
Ellipsoid	Solvability of an LMI	Checked via SDP
Arbitrary compact set	$\max_{p\in P}\min_{y\in\Delta}\{b(p)^T y : A(p)^T y=0\}\ge 0$	General theoretical result

Applications

Robust Control Theory

Systems of parametric linear inequalities are widely used in robust control theory. Parameter-dependent linear matrix inequalities (PLMIs) frequently arise in the analysis and synthesis of robust control systems.

Unlike linear matrix inequalities (LMIs), PLMIs are generally highly nonconvex and are therefore substantially more difficult to solve. Nevertheless, partial convexity properties of PLMIs can be exploited for optimization purposes. Based on these properties, relaxation methods have been developed to transform PLMI problems into standard LMI problems.

Two-Stage Robust Linear Programming

In two-stage robust linear programming, solvability conditions for systems of parametric linear inequalities play a central role. Chuong and Jeyakumar developed a generalized version of Farkas' lemma for systems of parametric linear inequalities with affinely adjustable variables, which makes it possible to establish strong duality results for two-stage robust linear programming.

In addition to establishing equality between the values of the primal and dual problems, this result makes it possible to compute the value of a two-stage robust linear programming problem by means of a semidefinite linear program. In the case of an ellipsoidal uncertainty set, a duality result in the form of a second-order cone program is obtained.

Stability Analysis of Parametric Systems

Stability analysis of systems of parametric linear inequalities is also of significant practical importance. Analyzing the relationships between systems of linear inequalities in different parametric settings, as well as studying continuity properties of mappings of the solution sets of parametric systems, provides a deeper understanding of the dependence of optimization problems on parameters.

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Conclusion

Determining conditions under which a parameter-dependent system of linear inequalities admits a solution for all values in a prescribed parameter domain is an important problem in optimization theory. As shown in this paper:

Farkas' lemma is a fundamental tool for characterizing infeasibility conditions of parametric systems of linear inequalities, and its parametric generalizations make it possible to establish strong duality results in robust optimization problems.

The **reduction to a maximization problem** allows the solvability condition of a parametric system over P to be expressed in optimization form. This criterion provides an effective means of determining whether there exist parameter values for which the system is infeasible.

The **connection with parametric linear programming** strengthens the theoretical foundation of the proposed approach and provides additional opportunities for practical applications.

The **practical applications** — including robust control theory, two-stage robust linear programming, and stability analysis of parametric systems — demonstrate the relevance of these theoretical results in engineering and economics.

Future research should focus on developing efficient algorithms for determining solvability conditions of parametric systems of linear inequalities, particularly for large-scale systems and complex parameter domains.

References

1. Tuan, H.D., Apkarian, P., Tuy, H. Advanced global optimization algorithms for parameterized LMIs. Proceedings of the 38th IEEE Conference on Decision and Control, 1999, pp. 310.
2. Keerthi, S.S., Sridharan, K. Solution of parametrized linear inequalities by Fourier elimination and its applications. Journal of Optimization Theory and Applications, Vol. 65, No. 1, pp. 161–169, 1990.
3. Chuong, T.D., Jeyakumar, V. Generalized Farkas lemma with adjustable variables and two-stage robust linear programs. Journal of Optimization Theory and Applications, Vol. 187, No. 2, pp. 488–519, 2020.
4. Cánovas, M.J., López, M.A., Parra, J. On the equivalence of parametric contexts for linear inequality systems. Journal of Computational and Applied Mathematics, Vol. 217, No. 2, pp. 448–456, 2008.
5. Ben-Tal, A., El Ghaoui, L., Nemirovski, A. Robust Optimization. Princeton University Press, 2009.
6. Rockafellar, R.T. Convex Analysis. Princeton University Press, 1970.