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RELATIONSHIP BETWEEN SYMMETRIC AND ANTISYMMETRIC POLYNOMIALS

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Abstract. This article examines the fundamental relationship between symmetric and antisymmetric (or alternating) polynomials in *n* variables. Symmetric polynomials are invariant under any permutation of the variables, while antisymmetric polynomials change sign under any transposition of the variables. The main connection between these two types of polynomials is manifested in their properties under multiplication: the product of two symmetric polynomials is symmetric; the product of a symmetric and an antisymmetric polynomial is antisymmetric; and the product of two antisymmetric polynomials is symmetric.

Keywords. Symmetric polynomials, antisymmetric polynomials, alternating polynomials, Vandermonde polynomial, symmetric functions, superalgebra, module, symmetric group, polynomial ring.

Introduction. Symmetric and antisymmetric polynomials are a special class of polynomials that occupy an important place in algebra and combinatorics. They possess certain properties with respect to permutations of variables, and these properties determine their interrelationship and role in the general theory of polynomials. Understanding this relationship helps to gain a deeper insight into the structure of polynomial rings, as well as to solve complex problems in fields such as Galois theory, invariant theory, and algebraic geometry. While symmetric polynomials remain invariant under any permutation of the variables, antisymmetric polynomials change their sign depending on the sign of the permutation. This fundamental difference manifests itself in their mutual interaction and in how they combine to represent general polynomials. The relationships between them are not only of theoretical importance but are also applied in practical computations, for example, in problems related to the roots of algebraic equations. In this section, we will examine in detail the interrelationship between symmetric and antisymmetric polynomials, their place in the general polynomial ring, and the key theorems that connect them. This analysis will serve as a foundation for understanding their mathematical structure and how they operate in various algebraic contexts [1].

Preliminaries. Any polynomial $P(x_1, \quad ,x_n)$ in n variables can be uniquely expressed as the sum of a symmetric and an antisymmetric part. This important property plays a central role in understanding the structure of polynomial rings. If a polynomial P is given, its symmetric part P_s and antisymmetric part P_a are defined by the following formulas. This decomposition allows any polynomial to be analyzed with respect to its behavior under permutations of variables. The symmetric part represents the component that remains unchanged under all permutations, while the antisymmetric part represents the component that changes sign only under odd permutations. This approach helps to simplify many problems in polynomial theory, since complex polynomials can be studied by splitting them into these two special cases. This decomposition allows the polynomial ring to be viewed as the direct sum of the space of symmetric polynomials and the space of antisymmetric polynomials, leading to a deeper understanding of their internal structure. This property is also closely related to other areas of polynomial theory, such as invariant theory, where symmetric polynomials play an important role as invariants [2].

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$$P_s(x_1, \quad, x_n) = \frac{1}{2}\left(P(x_1, \quad, x_n) + P(x_2, x_1, \quad, x_n)\right)$$

Symmetric part of a polynomial in two variables.

$$P_a(x_1, \quad, x_n) = \frac{1}{2}\left(P(x_1, \quad, x_n) - P(x_2, x_1, \quad, x_n)\right)$$

Antisymmetric part of a polynomial in two variables.

Results and discussion. In the general case, for a polynomial in n variables, determining the symmetric and antisymmetric parts requires summing over all \$n!\$ permutations. If S_n is the symmetric group and $\sigma \quad S_n$ is a permutation, we denote

$$P^\sigma(x_1, \quad, x_n) = P(x_{\sigma(1)}, \quad, x_{\sigma(n)}),$$

and $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation. In this case, the symmetric and antisymmetric parts of the polynomial P are defined as follows. These formulas provide a general mechanism for decomposing a polynomial into its symmetric and antisymmetric components. This decomposition is unique and is considered a fundamental result in polynomial theory. It allows the polynomial ring $K[x_1, \quad, x_n]$ to be represented as the direct sum of the ring of symmetric polynomials $K[x_1, \quad, x_n]^{S_n}$ and the space of antisymmetric polynomials A_n. This aids in a deeper understanding of the algebraic structure of the polynomial ring. This decomposition is also widely used in various areas of polynomial theory, such as invariant theory and algebraic combinatorics. It helps to simplify complex algebraic expressions and to determine their properties. This is particularly useful in computations involving polynomials and in proving theorems about their roots.

$$P_s(x_1, \quad, x_n) = \frac{1}{n!} \sum_{\sigma \quad S_n} P^\sigma(x_1, \quad, x_n)$$

Symmetric part of a general polynomial in n variables.

$$P_a(x_1, \quad, x_n) = \frac{1}{n!} \sum_{\sigma \quad S_n} \mathrm{sgn}(\sigma) P^\sigma(x_1, \quad, x_n)$$

Antisymmetric part of a general polynomial in n variables.

One of the most important connections between symmetric and antisymmetric polynomials is the relationship of antisymmetric polynomials with the Vandermonde determinant. The Vandermonde determinant is itself an antisymmetric polynomial and serves as the fundamental building block for all antisymmetric polynomials. More precisely, any antisymmetric polynomial can be uniquely expressed as the product of the Vandermonde determinant and a symmetric polynomial. This theorem completely characterizes the structure of antisymmetric polynomials and significantly simplifies their study. The Vandermonde determinant $\Delta(x_1, \quad, x_n)$ is given by the following formula. This determinant changes sign when any two variables are interchanged, which confirms its antisymmetric property. This property also follows from the fact that it vanishes when $x_i = x_j$. This theorem is of great importance in the theory of algebraic equations, particularly in the study of the discriminant, since the discriminant is expressed as the square of the Vandermonde determinant of the roots and is a symmetric polynomial. This connection delves into the deeper layers of polynomial theory and reveals the interrelations between algebraic structures.

The fundamental theorem concerning the Vandermonde determinant states that if $A(x_1, \quad, x_n)$ is an antisymmetric polynomial, then there exists a unique symmetric polynomial $S(x_1, \quad, x_n)$ such that

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$$A(x_1, \quad, x_n) = S(x_1, \quad, x_n)\ \Delta(x_1, \quad, x_n).$$

This theorem completely describes the structure of the ring of antisymmetric polynomials. Its proof is based on showing that the polynomial $A(x_1, \quad, x_n)$ is divisible by the factors $x_i - x_j$. Since A is antisymmetric, it vanishes when $x_i = x_j$, which means the polynomial is divisible by $x_i - x_j$. The product of all such factors is precisely the Vandermonde determinant. Consequently, the polynomial obtained by dividing A by Δ must be symmetric. This theorem is very important in algebraic geometry, particularly in studying ideals of polynomial rings, and in invariant theory. It allows the properties of antisymmetric polynomials to be analyzed by reducing them to the properties of symmetric polynomials. This helps to simplify complex algebraic problems and to find their solutions. This relationship is one of the fundamental pillars of polynomial theory and has a profound and wide-ranging scope of application [3-6].

The product of symmetric and antisymmetric polynomials also demonstrates their interrelationship. The product of two symmetric polynomials is always a symmetric polynomial. The product of two antisymmetric polynomials is always a symmetric polynomial, because if each antisymmetric polynomial changes sign under a permutation, their product undergoes a sign change twice and, as a result, the sign remains unchanged. For example, if A_1 and A_2 are antisymmetric, then

$$\sigma(A_1 A_2) = (\mathrm{sgn}(\sigma) A_1)(\mathrm{sgn}(\sigma) A_2) = (\mathrm{sgn}(\sigma))^2 A_1 A_2 = A_1 A_2.$$

This property is important for understanding the algebraic structure of the space of antisymmetric polynomials. Furthermore, the product of a symmetric polynomial and an antisymmetric polynomial is always an antisymmetric polynomial. These properties determine the interaction of different classes within the polynomial ring and their behavior under algebraic operations. These relationships are widely applied in various areas of polynomial theory, such as invariant theory and algebraic combinatorics. They help to simplify complex algebraic expressions and to determine their properties. This is particularly useful in computations involving polynomials and in proving theorems about their roots. These relationships are among the fundamental pillars of polynomial theory and have a profound and wide-ranging scope of application.

Elementary symmetric polynomials serve as generators for the ring of symmetric polynomials. Any symmetric polynomial can be uniquely expressed in terms of elementary symmetric polynomials. This fundamental theorem forms the basis of the theory of symmetric polynomials. The elementary symmetric polynomials $e_k(x_1, \quad, x_n)$ are defined as follows:

$$e_1 = x_1 + x_2 + \quad + x_n, \quad e_2 = \sum_{1 \le i < j \le n} x_i x_j, \quad , \quad e_n = x_1 x_2 \quad x_n.$$

These elementary polynomials completely describe the algebraic structure of the ring of symmetric polynomials. The fact that any symmetric polynomial can be expressed through them significantly simplifies working with symmetric polynomials. From the standpoint of the relationship with antisymmetric polynomials, the Vandermonde determinant can also be expressed in terms of elementary symmetric polynomials, although this expression is more complex. For example, in the case $n = 3$, $\Delta(x_1, x_2, x_3)$ can be expressed through elementary symmetric polynomials. This demonstrates an indirect but strong connection between symmetric and antisymmetric polynomials. These relationships are important in studying the connections between roots and coefficients of algebraic equations, particularly in generalizing Vieta's formulas. Furthermore, these relationships are also widely used in invariant theory and algebraic combinatorics, where symmetric and antisymmetric

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polynomials play a key role in constructing invariants of various algebraic structures. This further increases their theoretical and practical significance.

The interrelationship between symmetric and antisymmetric polynomials has left a deep mark on various areas of polynomial theory, including algebraic number theory and algebraic geometry. For example, in Galois theory, the discriminant of a polynomial is defined as the square of the Vandermonde determinant of its roots and is always a symmetric polynomial. The discriminant plays an important role in determining the relationships between the roots of a polynomial, in particular, whether any of them are repeated. If the discriminant is equal to zero, then the polynomial has at least two equal roots. This property is very important in solving algebraic equations and analyzing their properties. Furthermore, these polynomials are also of great significance in invariant theory. Invariant theory studies polynomials that remain unchanged under transformations from a certain group acting on the variables. While symmetric polynomials are invariants of the symmetric group S_n, antisymmetric polynomials are invariants under a certain character of the group S_n. These relationships reveal the deep symmetries of algebraic structures and simplify their analysis. This is of great importance in modern mathematical research, particularly in algebraic topology and algebraic combinatorics, where it aids in understanding the properties of complex mathematical objects.

Moreover, the relationships between symmetric and antisymmetric polynomials are also widely applied in algebraic combinatorics. For example, in the theory of Schur polynomials, they form a special basis of symmetric polynomials and are closely related to Young diagrams. Schur polynomials play an important role in representation theory and combinatorics. Antisymmetric polynomials, in turn, are used in studying other combinatorial objects related to specific types of fillings of Young diagrams. These relationships demonstrate the deep connections between polynomial theory and combinatorics. They allow complex combinatorial problems to be solved using algebraic methods. For instance, the theory of symmetric and antisymmetric polynomials is very useful in counting Young diagrams or determining their properties. This is also reflected in various areas of mathematical research, particularly in applied sciences such as physics and computer science. The mutual interaction of these polynomials serves as a primary tool for understanding the internal structure of mathematical objects and their behavior under various transformations. This further increases their theoretical and practical significance and plays an important role in modern mathematical research.

The connection between symmetric and antisymmetric polynomials can also be related to their derivatives and integrals. The derivative of a symmetric polynomial is generally not symmetric, but it can be decomposed into symmetric and antisymmetric parts. Similarly, the derivative of an antisymmetric polynomial is generally not antisymmetric. However, by analyzing the symmetric and antisymmetric components of the polynomials resulting from these operations, information about their original properties can be obtained. This is important in differential algebra and algebraic analysis. For example, applying the Laplace operator to a symmetric polynomial yields a symmetric polynomial, which shows that the space of symmetric polynomials is closed under the Laplace operator. This property is very important in mathematical physics, particularly in studying symmetric systems in quantum mechanics and statistical mechanics. Similar properties can also be studied for antisymmetric polynomials. This further increases their theoretical and practical significance and plays an important role in modern mathematical research. These relationships are among the fundamental pillars of polynomial theory and have a profound and wide-ranging scope of application.

Conclusion. In conclusion, the relationship between symmetric and antisymmetric polynomials is one of the central topics in polynomial theory. The fact that any polynomial can be uniquely represented as the sum of symmetric and antisymmetric parts, as well as the representation of antisymmetric polynomials as the product of the Vandermonde determinant and a symmetric

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polynomial, demonstrate the deep algebraic relations between these two classes of polynomials. These relationships are not only of theoretical importance but are also widely applied in various mathematical fields such as the theory of algebraic equations, invariant theory, Galois theory, and algebraic combinatorics. Their mutual interaction helps to gain a deeper understanding of the structure of polynomial rings and to solve complex algebraic problems. This is also reflected in modern mathematical research, particularly in applied sciences such as physics and computer science. The mutual interaction of these polynomials serves as a primary tool for understanding the internal structure of mathematical objects and their behavior under various transformations. This further increases their theoretical and practical significance and plays an important role in modern mathematical research. These relationships are among the fundamental pillars of polynomial theory and have a profound and wide-ranging scope of application.

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