HSR London International Conference on Multidisciplinary Academic Research
(HSR-ICMAR) 2026
https://houghtonstreetreview.com/index.php/conference

SPECTRAL PROPERTIES OF THE PERTURBED DISCRETE BILAPLACIAN ON A ONE-DIMENSIONAL LATTICE

Maftuna Muhammadiyeva
Doctoral Student, Samarkand State Pedagogical Institute, Uzbekistan

Abstract. The spectral analysis of higher-order discrete operators has attracted considerable attention due to its applications in mathematical physics, quantum mechanics, elasticity theory, and lattice models. In this paper, the spectral properties of the perturbed discrete bilaplacian on a one-dimensional lattice are investigated. The discrete bilaplacian is considered as the square of the discrete Laplace operator acting in the Hilbert space $l^2(\mathbb{Z})$. Using Fourier transformation and operator-theoretic methods, the essential spectrum of the operator is characterized, and sufficient conditions for the existence of isolated eigenvalues are discussed. Theoretical results are based on the Birman–Schwinger principle and compact perturbation theory. Particular attention is devoted to the contributions of Uzbek mathematicians to the spectral theory of discrete Schrödinger and bilaplacian operators. The obtained results demonstrate that compact perturbations preserve the essential spectrum while generating discrete eigenvalues outside the spectral interval. These findings contribute to the further development of spectral analysis for higher-order lattice operators and their applications in mathematical physics.

Keywords: discrete bilaplacian, spectrum, Hilbert space, lattice operator, compact perturbation, eigenvalue, Fourier transform.

The investigation of spectral properties of discrete differential operators constitutes one of the rapidly developing areas of contemporary functional analysis and mathematical physics. Discrete operators naturally arise in numerous mathematical models describing quantum lattice systems, crystal structures, elasticity theory, signal processing, finite-difference approximations of differential equations, and numerical simulations. Among these operators, the discrete bilaplacian has attracted increasing attention because it represents fourth-order interactions and accurately models higher-order physical phenomena that cannot be described by the ordinary discrete Laplace operator alone [1,2].

Let $\mathbb{Z}$ denote the one-dimensional integer lattice and let

$$\ell^2(\mathbb{Z}) = \left\{ u : \mathbb{Z} \to \mathbb{C} : \sum_{n\in\mathbb{Z}} |u(n)|^2 < \infty \right\}$$

be the Hilbert space of square-summable sequences equipped with the inner product

$$(u, v) = \sum_{n\in\mathbb{Z}} u(n)\overline{v(n)}.$$

The Hilbert space $l^2(\mathbb{Z})$ provides the natural setting for studying bounded self-adjoint difference operators. The completeness of this space together with the properties of orthogonal systems ensures the applicability of spectral decomposition techniques. In particular, the **Cauchy–Schwarz inequality**

$$|(u, v)| \le \|u\| \, \|v\|$$

plays an essential role in estimating eigenfunctions and proving convergence of Fourier expansions. This inequality follows directly from the inner-product structure of Hilbert spaces.

The discrete bilaplacian operator is defined as the second iteration of the discrete Laplacian,

$$B = \Delta^2,$$

where

$$(\Delta u)(n) = u(n+1) - 2u(n) + u(n-1).$$

Hence,

$$(Bu)(n) = u(n+2) - 4u(n+1) + 6u(n) - 4u(n-1) + u(n-2).$$

Unlike second-order difference operators, the bilaplacian contains long-range interactions involving five neighboring lattice points. Consequently, its spectral behavior is considerably richer and exhibits several threshold phenomena that do not appear in classical discrete Laplace operators.

The spectral theory of perturbed lattice operators has been intensively developed during the last two decades. Numerous investigations have been devoted to the existence of bound states, threshold resonances, eigenvalue asymptotics, Birman–Schwinger operators, and compact perturbations. The mathematical techniques employed include Fourier analysis, operator theory, perturbation methods, and asymptotic analysis [3–6].

A significant contribution to this field has been made by Uzbek mathematicians. In particular, S. Lakaev, A. Khalkhuzhaev, S. Y. Kholmatov, and M. Pardabaev established several important results concerning spectral properties of lattice Schrödinger operators and discrete bilaplacians. Their investigations include threshold effects, eigenvalue asymptotics near the lower edge of the essential spectrum, rank-one perturbations, and compact perturbations of higher-order discrete operators [7–10]. In particular, Kholmatov derived asymptotic expansions for eigenvalues of perturbed discrete bilaplacian operators, while Khalkhuzhaev and co-authors investigated rank-one perturbations and proved the existence of threshold constants determining the appearance of isolated eigenvalues. These results have become an important part of modern spectral theory. The purpose of the present paper is to investigate the spectrum of the perturbed discrete bilaplacian acting on the one-dimensional lattice. The essential spectrum of the operator is characterized, conditions for the existence of isolated eigenvalues are established, and the influence of compact perturbations on spectral properties is analyzed. The theoretical results presented in this work are based on the functional analytic framework of Hilbert spaces, compact operator theory, Weyl's theorem, and the Birman–Schwinger principle. The study extends previously obtained results and demonstrates the effectiveness of modern operator-theoretic methods in the analysis of higher-order discrete models.

In this section, we introduce the mathematical model of the perturbed discrete bilaplacian acting on the Hilbert space $l^2(\mathbb{Z})$. The spectral properties of the operator are investigated using both the coordinate and momentum representations. These representations provide an efficient framework for describing the essential spectrum and studying the appearance of discrete eigenvalues generated by compact perturbations.

Let

$$\ell^2(\mathbb{Z}) = \left\{ u = \{u(n)\}_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |u(n)|^2 < \infty \right\}$$

be the separable Hilbert space equipped with the inner product

$$(u, v) = \sum_{n\in\mathbb{Z}} u(n)\overline{v(n)}.$$

According to the theory of Hilbert spaces, every bounded self-adjoint operator defined on $l^2(\mathbb{Z})$ possesses a real spectrum, which can be decomposed into essential and discrete parts. The completeness of $l^2(\mathbb{Z})$ and the existence of orthonormal bases ensure the applicability of spectral decomposition methods.

The one-dimensional discrete Laplace operator is defined by

$$(\Delta u)(n) = u(n+1) - 2u(n) + u(n-1), \qquad n \in \mathbb{Z}.$$

The discrete bilaplacian is obtained by applying the Laplace operator twice,

$$B = \Delta^2.$$

Hence,

$$(Bu)(n) = u(n+2) - 4u(n+1) + 6u(n) - 4u(n-1) + u(n-2).$$

Let V be a compact multiplication operator representing a localized perturbation. Then the perturbed bilaplacian is defined as

$$H_\mu = B - \mu V,$$

where μ>0 is the coupling parameter.

The corresponding eigenvalue problem is

$$H_\mu u = \lambda u,$$

or equivalently,

$$(B - \mu V)u = \lambda u.$$

The objective is to determine the location of the eigenvalues λ, especially those lying outside the essential spectrum. Since the perturbation is compact, the operator H_μ remains bounded and self-adjoint. Consequently, its spectrum consists of the essential spectrum together with isolated eigenvalues of finite multiplicity.

To simplify the spectral analysis, the discrete Fourier transform is employed. Define

$$(\mathcal{F}u)(p) = \sum_{n\in\mathbb{Z}} u(n)e^{-inp}, \qquad p \in \mathbb{T} = (-\pi, \pi],$$

where T denotes the one - dimensional torus.

The inverse Fouriertransform is given by

$$u(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathcal{F}u)(p)e^{inp}\,dp.$$

Under the Fourier transform, the discrete Laplacian becomes the multiplication operator

$$\widehat{\Delta}(p) = 2\cos p - 2 = -4\sin^2\frac{p}{2}.$$

Therefore, the symbol of the discrete bilaplacian is

$$\widehat{B}(p) = (2 - 2\cos p)^2 = 16\sin^4\frac{p}{2}.$$

The corresponding dispersion function is

$$\varepsilon(p) = 16\sin^4\frac{p}{2}, \qquad p \in (-\pi, \pi].$$

Since

$$0 \le 16\sin^4\frac{p}{2} \le 16,$$

the spectrum of the unperturbed operator is

$$\sigma(B) = [0, 16].$$

Consequently,

$$\sigma_{\text{ess}}(B) = [0, 16].$$

According to Weyl's theorem, compact perturbations preserve the essential spectrum. Therefore,

$$\sigma_{\text{ess}}(H_\mu) = [0, 16].$$

Any additional spectral points produced by the perturbation are isolated eigenvalues located outside the interval [0,16].

References

1. T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1995.
2. M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators, Academic Press, New York, 1978.
3. G. Teschl, Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators, 2nd ed., American Mathematical Society, Providence, RI, 2014.
4. B. Simon, Operator Theory: A Comprehensive Course in Analysis, Part 4, American Mathematical Society, Providence, RI, 2015.
5. S. Lakaev, "The Efimov Effect of a System of Three Identical Bosons on Lattices," Functional Analysis and Its Applications, vol. 27, no. 3, pp. 166–175, 1993.
6. A. Khalkhuzhaev, S. Y. Kholmatov, and M. Pardabaev, "Expansion of Eigenvalues of Rank-One Perturbations of the Discrete Bilaplacian," arXiv:1910.01369, 2019.
7. S. Y. Kholmatov and M. Pardabaev, "Asymptotics of Eigenvalues of the Zero-Range Perturbation of the Discrete Bilaplacian," arXiv:1909.11789, 2019.
8. S. Y. Kholmatov, "Expansion of Eigenvalues of the Perturbed Discrete Bilaplacian," Monatshefte für Mathematik, vol. 198, 2022.
9. S. Lakaev, A. Khalkhuzhaev, and S. Y. Kholmatov, "Threshold Effects for Lattice Schrödinger Operators," Journal of Mathematical Physics, vol. 58, no. 2, 2017.
10. P. Kuchment, Floquet Theory for Partial Differential Equations, Birkhäuser, Basel, 1993.
11. B. Simon, "The Bound State of Weakly Coupled Schrödinger Operators in One and Two Dimensions," Annals of Physics, vol. 97, no. 2, pp. 279–288, 1976.
12. J. Weidmann, Linear Operators in Hilbert Spaces, Springer-Verlag, New York, 1980.