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DISPERSION EQUATION FOR A CYLINDRICAL ROD IN A VISCOELASTIC HALF-SPACE: ACCOUNTING FOR THE EFFECT OF THE FREE BOUNDARY

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Abstract. In this paper, a dispersion equation is derived for harmonic waves propagating along a cylindrical rod embedded in a viscoelastic half-space parallel to its free surface. The rheological properties of the medium are described by Boltzmann–Volterra hereditary operators with the Rzhanitsyn–Koltunov kernel. The waves reflected from the free boundary are re-expanded about the rod center using the plane-wave expansion and Graf's addition theorem. As a result, the problem is reduced to an infinite system of algebraic equations, the vanishing of whose determinant yields the dispersion equation. The complex roots are found by Muller's method, and an algorithm for checking their completeness by means of the argument principle is proposed. Calculations performed in the fundamental-harmonic approximation show that the free surface reduces the frequency of the first torsional mode by up to 1.05% and increases its attenuation by up to 78%.

Keywords: viscoelastic half-space, cylindrical rod, dispersion equation, hereditary kernel, Graf's addition theorem, complex frequency, Muller's method.

Introduction

The analysis of underground pipelines, pile foundations and tunnel structures often employs the model of a cylindrical rod embedded in soil. To assess their response to seismic and man-made dynamic loads, the dispersion of waves in such a system must be known. Soils and polymer media exhibit pronounced rheological properties, whereas a purely elastic model accounts neither for wave attenuation nor for the additional dispersion associated with rheology.

The Pochhammer–Chree dispersion equation for waves in a free rod and the main results of the theory of elastic waveguides are summarized in the monograph [1]. Modern numerical methods for computing trapped and leaky modes in waveguides embedded in an unbounded medium were developed in [2, 3], while the propagation and attenuation of waves, including the torsional wave, in pipes buried in soil or embedded in concrete were studied in [4, 5]. The foundations of the theory of hereditary viscoelasticity are presented in the monograph [6]; the propagation of natural waves in cylindrical cavities, shells and multilayer cylindrical bodies in a viscoelastic medium, based on the Boltzmann–Volterra model with the Rzhanitsyn–Koltunov kernel, was investigated in [7–11]. However, most of these studies assume the surrounding medium to be unbounded, i.e., symmetric about the rod. When the medium is a half-space, the waves reflected from the free surface interact with the rod and the symmetry is broken. The problem in which hereditary viscoelasticity, the free

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boundary and wave propagation along the rod are considered together has not been sufficiently studied.

The aim of this study is to derive a dispersion equation for a cylindrical rod in a viscoelastic half-space that rigorously accounts for the effect of the free boundary, and to develop a reliable numerical algorithm for finding its complex roots.

Problem statement

A viscoelastic medium occupies the half-space $y\le 0$. It contains an elastic rod whose axis is parallel to the Oz axis, with its center at the point $(0,-h)$ and radius R ($h>R$). The rod material is characterized by the parameters λ_1,μ_1,ρ_1 (Lamé parameters and density), while the density of the medium is ρ and its instantaneous Lamé parameters are λ_0,μ_0. The constitutive equations of the medium are taken in the Boltzmann–Volterra form:

$$\sigma_{ij}=\tilde{\lambda}\,\varepsilon_{kk}\,\delta_{ij}+2\tilde{\mu}\,\varepsilon_{ij},\qquad \tilde{\lambda}f(t)=\lambda_0\left[f(t)-\int_{-\infty}^{t}Q_\lambda(t-\tau)\,f(\tau)\,d\tau\right],$$

where ε_{kk} is the volumetric strain; the operator $\tilde{\mu}$ is defined similarly with the kernel Q_μ. In what follows, we set $Q_\lambda=Q_\mu=Q$ and choose the Rzhanitsyn–Koltunov kernel [8–11] as the relaxation kernel:

$$Q(t)=A\,e^{-\beta t}\,t^{\alpha-1},\qquad 0<\alpha<1,\quad A>0,\quad \beta>0.$$

The solution is sought in the form $\exp\big(i(kz-\omega t)\big)$. The hereditary operators then reduce to complex moduli:

$$\tilde{\lambda}(\omega)=\lambda_0\left[1-\frac{A\,\Gamma(\alpha)}{(\beta-i\omega)^\alpha}\right],\qquad \tilde{\mu}(\omega)=\mu_0\left[1-\frac{A\,\Gamma(\alpha)}{(\beta-i\omega)^\alpha}\right],$$

where $\Gamma(\alpha)$ is the Euler gamma function. The integral representation converges in the region $\mathrm{Im}\,\omega>-\beta$; the resulting expression is analytically continued (using the principal branch of the power function) to the entire complex plane except for a cut running downward along the imaginary axis from the branch point $\omega=-i\beta$. Therefore, the complex moduli, and hence the dispersion equation, are analytic functions of ω, which makes it possible to construct a dispersion equation with complex frequency. The motion of the medium and the rod is governed by the Lamé equations (for the rod, the hereditary operators are replaced by the constant parameters λ_1,μ_1,ρ_1):

$$\tilde{\mu}\,\Delta\mathbf{u}+\left(\tilde{\lambda}+\tilde{\mu}\right)\nabla\left(\nabla\cdot\mathbf{u}\right)=\rho\,\frac{\partial^2\mathbf{u}}{\partial t^2}.$$

The boundary conditions are as follows:

1. $y=0$ (free surface): $\sigma_{yy}=\sigma_{xy}=\sigma_{yz}=0$.
2. $r=R$ (contact surface, perfect bonding): $\mathbf{u}^{(1)}=\mathbf{u}^{(2)}$ and continuity of the stresses $\sigma_{rr},\sigma_{r\theta},\sigma_{rz}$.
3. $x^2+y^2\to\infty$, $y\le 0$: radiation condition — waves in the medium propagate away from the rod or decay.

Solution method

The displacements are expressed in terms of three scalar potentials: $\mathbf{u}=\nabla\varphi+\nabla\times(\psi\mathbf{e}_z)+\nabla\times\nabla\times(\chi\mathbf{e}_z)$. In the cross-section, each potential satisfies the Helmholtz equation:

$$\Delta_\perp\varphi+p^2\varphi=0,\quad \Delta_\perp\{\psi,\chi\}+q^2\{\psi,\chi\}=0,\qquad p^2=\frac{\rho\,\omega^2}{\tilde{\lambda}+2\tilde{\mu}}-k^2,\quad q^2=\frac{\rho\,\omega^2}{\tilde{\mu}}-k^2.$$

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Inside the rod, the solution is sought in terms of regular functions, e.g., $\varphi_1=\sum_n a_n J_n(p_1 r)\, e^{in\theta}$. The field in the medium consists of two parts: the waves scattered by the rod, $\sum_n S_n H_n^{(1)}(pr)\, e^{in\theta}$, and the waves reflected from the free surface.

To construct the reflected field, the Hankel functions are expanded into a plane-wave integral [12]. Each plane wave is reflected from the surface $y=0$ with the reflection coefficients $\mathscr{R}_{\eta\zeta}(\xi)$, where the indices $\eta,\zeta\in\{p,q\}$ denote the type (longitudinal or transverse) of the incident and reflected waves, i.e., mode conversion is taken into account. The reflected plane waves are then re-expanded into a series of Bessel functions about the rod center using the Jacobi–Anger formula and Graf's addition theorem [12]. As a result, the coefficients of the reflected field are linearly related to those of the scattered field: $\mathbf{B}=K(\omega,k,h)\,\mathbf{S}$, where

$$K_{mn}^{\eta\zeta}=\frac{1}{\pi}\int_{-\infty}^{\infty}\mathscr{R}_{\eta\zeta}(\xi)\, G_{mn}^{\eta\zeta}(\xi)\, e^{i(\gamma_\eta+\gamma_\zeta)h}\, d\xi, \qquad \gamma_\eta=\sqrt{\eta^2-\xi^2}, \quad \operatorname{Im}\gamma_\eta\geq 0.$$

Here $G_{mn}^{\eta\zeta}$ are known functions composed of angular factors. Writing the contact conditions at $r=R$ for each harmonic and eliminating the coefficients inside the rod, we obtain an infinite homogeneous system $(D+EK)\,\mathbf{X}=0$. The condition for this system to have a nontrivial solution is the sought dispersion equation:

$$\Delta(\omega,k;h)=\det[D(\omega,k)+E(\omega,k)\,K(\omega,k,h)]_{|m|,|n|\leq N}=0.$$

$h\to\infty$ gives $K\to 0$, and the equation reduces to the known equation $\prod_n \det D_n=0$ for a rod in an unbounded medium, while $A\to 0$ yields the case of an elastic half-space. These limiting cases are used to verify the results.

Numerical algorithm. For the trapped modes under consideration, i.e., those decaying exponentially in the medium ($\omega/k<c_R$, where c_R is the Rayleigh wave velocity), the branch points $\pm p,\pm q$ and the Rayleigh poles of the reflection coefficients do not lie on the real axis, and viscosity shifts them further into the complex plane. Therefore, the integrands of the K_{mn} integrals are regular on the real axis and decay exponentially as $|\xi|\to\infty$; these integrals are evaluated by high-order Gaussian quadrature. For a given real k, the complex roots of $\Delta=0$ are found by Muller's method [13]:

$$\omega_{j+1}=\omega_j-\frac{2\,\Delta(\omega_j)}{b_j\pm\sqrt{b_j^2-4\,a_j\,\Delta(\omega_j)}},$$

where a_j,b_j are the coefficients of the parabola passing through the points $\omega_{j-2},\omega_{j-1},\omega_j$; the sign is chosen so that the denominator has the larger modulus. The initial approximations are taken from the roots for $h\to\infty$ and are continued step by step in k. The completeness of the roots found is checked by the argument principle [14]: for a given contour C, the number of roots inside it is

$$N_0=\frac{1}{2\pi i}\oint_C \frac{\Delta'(\omega)}{\Delta(\omega)}\, d\omega$$

where the contour C must not cross the branch points of the function Δ ($\omega=-i\beta$, as well as the points corresponding to the conditions $p=0$ and $q=0$) or their cuts. In

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general, the truncation order N of the infinite system is increased until the relative change in the roots is less than 10^{-6}; the calculations presented below, however, were performed in the first (N = 0) approximation.

Results and discussion

The calculations were performed in dimensionless variables: $\Omega=\omega R/c_s$, $\kappa=kR$, $H=h/R$, with time measured in units of R/c_s (the rheological parameters A and β are nondimensionalized in the same units), where $c_s=\sqrt{\frac{\mu_0}{\rho}}$ is the instantaneous shear wave velocity in the medium. Parameters: Poisson's ratio of the medium $\nu=0.25$; $c_{s1}/c_s=0.5$, $\rho_1/\rho=0.8$ (hence $\mu_1/\mu_0=0.2$); rheological parameters $A=0.048$, $\alpha=0.25$, $\beta=0.05$. For these values $\int_0^\infty Q(t)dt=A\Gamma(\alpha)\beta^{-\alpha}\approx0.37<1$ (0.62 and 0.38 for $\alpha=0.1$ and $\alpha=0.5$ in Table 2, respectively), i.e., the long-term modulus of the medium is positive. Since the rod is soft relative to the medium, the field of the first torsional mode decays exponentially in the radial direction in the medium (the mode is trapped). In an unbounded elastic medium, the cutoff point of this mode ($\Omega=\kappa$) corresponds to $\kappa\approx2$; in the half-space, however, the mode is non-radiating only for $\Omega/\kappa<c_R/c_s\approx0.919$, i.e., for $\kappa\gtrsim2.3$. Therefore, the calculations were performed in the range $\kappa\geq2.5$ for this mode (n=0) in the fundamental-harmonic approximation, i.e., with truncation order N = 0, retaining in the matrix K only the diagonal block K_{00} (which accounts for mode conversion between longitudinal and transverse waves). Since the integrand decays exponentially, the infinite integration interval was replaced by a finite interval [−L, L] whose truncation remainder is smaller than the computational accuracy; the integrals were evaluated by Gauss–Legendre quadrature with 1501 nodes, and doubling the number of nodes changed the result by less than 10^{-11}. Muller iterations were continued until a relative accuracy of 10^{-10} was reached.

Verification. For $A=0$, the roots found for the half-space remained real (the imaginary part is zero to machine precision). This corresponds to the absence of energy radiation by the trapped mode and indirectly confirms the correctness of the algorithm. As H increases, the roots tend to those for the unbounded medium: for $\kappa=3$, $H=5$, the difference is of the order of 10^{-8}.

Table 1. Complex frequency of the first torsional mode Ω

κ	Elastic, $H=\infty$	Viscoelastic, $H=\infty$	$H=2$	$H=1.5$	Δ Re Ω, %	Im Ω increase, %
2.5	2.1768	2.1507 − 0.01428i	2.1455 − 0.01929i	2.1281 − 0.02540i	−1.05	+78
3	2.3469	2.3292 − 0.00914i	2.3284 − 0.00944i	2.3229 − 0.01065i	−0.27	+16.5
4	2.7111	2.7006 − 0.00526i	2.7006 − 0.00527i	2.6995 − 0.00545i	−0.04	+3.6
5	3.1072	3.1002 − 0.00349i	3.1002 − 0.00349i	3.0999 − 0.00353i	−0.01	+1.1
6	3.5274	3.5224 − 0.00248i	3.5224 − 0.00248i	3.5223 − 0.00249i	−0.003	+0.4

The last two columns compare the case H=1.5 with the viscoelastic case H=∞.

Effect of viscosity. In an unbounded medium, the rheological properties reduce the frequency by 0.1–1.2% and give rise to attenuation. The attenuation decreases with increasing wavenumber: for $\kappa=2.5$, Im $\Omega=-0.0143$, whereas for $\kappa=6$ it is −0.0025. There are two reasons for this: first, at higher frequencies most of the wave energy is concentrated inside the elastic rod; second, for the Rzhanitsyn–Koltunov kernel, the imaginary part of the complex modulus decreases with increasing frequency.

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Effect of the free boundary. The free surface reduces the mode frequency and increases the attenuation. The effect grows sharply as the rod approaches the surface and at wavenumbers close to the cutoff: for $\kappa=2.5$ and $H=1.5$, $\mathrm{Re}\,\Omega$ decreased by 1.05% and the attenuation increased by 78%; for $\kappa\geq 5$, the frequency change does not exceed 0.01% and the increase in attenuation does not exceed 1.1%. This is consistent with the decay of the amplitude of the field returning to the rod according to the law $e^{-2|q|h}$: as κ increases, $|q|$ increases and the coupling with the surface weakens. In the elastic case the frequency also decreases (by 0.58% for $\kappa=2.5$, $H=1.5$), but no attenuation appears. Hence, the additional attenuation in the half-space results from the combined effect of the viscous medium and the free surface: the free surface weakens the localization of the mode and increases the fraction of energy in the viscous medium.

Table 2. Attenuation at $\kappa=3$ as a function of the singularity parameter α ($-\mathrm{Im}\,\Omega$)

α	$H=\infty$	$H=3$	$H=2$	$H=1.5$
0.1	0.04398	0.04432	0.05062	0.07224
0.25	0.00914	0.00915	0.00944	0.01065
0.5	0.00504	0.00504	0.00515	0.00566

α has a strong effect: its decrease, i.e., a stronger kernel singularity, sharply increases the attenuation: for $\alpha=0.1$, the attenuation is 8.7 times greater than for $\alpha=0.5$. The reason is that at the frequency $\Omega\approx 2.3$ the real part of the complex shear modulus for $\alpha = 0.1$, 0.25 and 0.5 is approximately $0.59\mu_0$, $0.87\mu_0$ and $0.96\mu_0$, respectively: at $\alpha = 0.1$ the medium softens appreciably, the mode approaches the cutoff frequency, and its field penetrates deeper into the medium. The effect of the free surface is also more pronounced for a strongly singular kernel: for $H=1.5$, the attenuation increases by 64% for $\alpha=0.1$ and by 12% for $\alpha=0.5$.

Conclusions

1. A dispersion equation has been derived for a cylindrical rod in a viscoelastic half-space that rigorously accounts for the waves reflected from the free boundary. It contains the rod depth $H=h/R$ as a new parameter.
2. Using closed-form expressions of the complex moduli for the Rzhanitsyn–Koltunov kernel, the dispersion determinant was treated as an analytic function of complex frequency (in the plane with a cut). This made it possible to apply Muller's method and the argument principle.
3. Calculations in the fundamental-harmonic approximation showed that the free surface reduces the frequency of the first torsional mode (by up to 1.05% at $H=1.5$, $\kappa=2.5$) and increases the attenuation (by up to 78%). The effect is greatest near the cutoff frequency and for a strongly singular kernel ($\alpha=0.1$); for $\kappa\geq 5$, the increase in attenuation does not exceed 1.1% and the frequency change does not exceed 0.01%. Assessing the contribution of higher ($n\neq 0$) harmonics when the rod lies very close to the surface is a task for future research.
4. The results obtained can be applied to the dynamic analysis of underground pipelines and pile foundations.

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