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SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR PARABOLIC AND ELLIPTIC EQUATIONS WITH THE BESSEL OPERATOR

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Abstract. The thesis studies boundary value problems for parabolic and elliptic equations containing the singular Bessel differential operator $B_k=\frac{\partial^2}{\partial x^2}+\frac{k}{x}\frac{\partial}{\partial x}$, $k>0$. For the first boundary value problem for the singular heat equation and for the Dirichlet problem for the Weinstein equation in a rectangle, uniqueness is proved by the energy method and the extremum principle, and solutions are constructed as Fourier–Bessel series. Sufficient conditions on the data that guarantee convergence are given. The Cauchy problem on the half-axis is solved by the Hankel transform, which yields an explicit kernel expressed through the modified Bessel function. The results generalize the classical formulas for radially symmetric solutions of the heat and Laplace equations to an arbitrary "dimension" $k>0$.

Keywords: Bessel operator, singular coefficient, heat equation, Weinstein equation, Fourier–Bessel series, Hankel transform, energy method, extremum principle, uniqueness, existence.

INTRODUCTION

Equations with the Bessel operator appear naturally whenever a problem has axial or spherical symmetry. For a radial function $u(r)$ in $\mathbb{R}^n$ the Laplace operator reduces to $u_{rr}+\frac{n-1}{r}u_r$, that is, to the operator B_k with $k=n-1$. Allowing an arbitrary real $k>0$ leads to the theory of equations of "fractional dimension", known as generalized axially symmetric potential theory, founded by A. Weinstein [3]. Similar equations describe heat conduction in cylinders and balls, diffusion in porous media and the radial part of wave processes [1, 7].

The main difficulty is that the coefficient k/x is unbounded as $x\to+0$, so the line $x=0$ is a line of degeneration of the equation. A classical boundary condition cannot be prescribed there arbitrarily; it is replaced by a condition of the form $x^k u_x\to 0$ as $x\to+0$, which selects bounded solutions. The systematic theory of such problems was developed by I.A. Kipriyanov in special weighted function spaces [2]; the harmonic-analysis tools (the Hankel transform and the generalized translation operator) go back to B.M. Levitan [4], and the transmutation method is presented in [6]. Boundary value problems for equations with singular coefficients have also been studied extensively by the Uzbek school of M.S. Salakhitdinov and A.K. Urinov [8].

The aim of this thesis is to give a unified and self-contained treatment of the model parabolic and elliptic problems with the Bessel operator: to prove uniqueness, construct the solutions explicitly, and compare the structure of the results.

MAIN PART

Notation. Throughout, $k>0$ and $\nu=\frac{k-1}{2}>-\frac{1}{2}$. The key identity, verified by direct differentiation, is

$$x^k B_k u=\frac{\partial}{\partial x}\left(x^k\frac{\partial u}{\partial x}\right), \qquad (1)$$

so B_k is a symmetric operator in the weighted space $L^2\left((0,l);x^k dx\right)$. Moreover, $B_k[x^{-\nu}J_\nu(\lambda x)]=-\lambda^2 x^{-\nu}J_\nu(\lambda x)$, where J_ν is the Bessel function of the first kind [1].

Problem P (parabolic). In the domain $\Omega=\{0<x<l,\ 0<t\le T\}$ find a function $u(x,t)$ satisfying

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$$u_t = u_{xx} + \frac{k}{x}u_x, \qquad (x,t)\in\Omega, \tag{2}$$

$$u(x,0)=\varphi(x), \qquad u(l,t)=0, \qquad x^k u_x(x,t)\to 0 \quad (x\to+0). \tag{3}$$

Uniqueness. Let u be a solution of problem P with $\varphi\equiv 0$. Multiplying equation (2) by $x^k u$, integrating over $(0,l)$ and using identity (1) together with conditions (3), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^l x^k u^2\,dx = -\int_0^l x^k u_x^2\,dx \le 0. \tag{4}$$

Hence the weighted energy $E(t)=\int_0^l x^k u^2 dx$ does not increase; since $E(0)=0$, we get $u\equiv 0$. Therefore problem P has at most one solution.

Existence. Separation of variables $u=X(x)T(t)$ leads to the singular Sturm–Liouville problem $B_k X+\lambda X=0$, $X(l)=0$, $x^k X'(x)\to 0$ as $x\to+0$. The second condition rules out the solution $x^{-\nu}Y_\nu\left(\sqrt{\lambda}x\right)$, so the eigenfunctions are $X_n=x^{-\nu}J_\nu(\mu_n x/l)$ with eigenvalues $\lambda_n=\mu_n^2/l^2$, where $0<\mu_1<\mu_2<\dots$ are the positive zeros of J_ν. They are orthogonal with weight x^k, and the solution is given by

$$u(x,t)=x^{-\nu}\sum_{n=1}^{\infty} c_n\, e^{-\mu_n^2 t/l^2} J_\nu\left(\frac{\mu_n x}{l}\right), \tag{5}$$

$$c_n=\frac{2}{l^2 J_{\nu+1}^2(\mu_n)}\int_0^l \xi^{\nu+1}\varphi(\xi) J_\nu\left(\frac{\mu_n \xi}{l}\right)d\xi. \tag{6}$$

Theorem 1. Let $\varphi\in C^1[0,l]$ and $\varphi(l)=0$. Then series (5) converges uniformly in $\overline{\Omega}$, may be differentiated termwise any number of times for $t\ge t_0>0$, and its sum is the unique classical solution of problem P.

The factor $e^{-\mu_n^2 t/l^2}$ shows that $u(x,t)\sim c_1 x^{-\nu}J_\nu(\mu_1 x/l)e^{-\mu_1^2 t/l^2}$ as $t\to\infty$. Since $\mu_1(\nu)$ increases with ν, the solution decays faster for larger k: for example, $\mu_1=2.4048$ for $k=1$ (cylinder) and $\mu_1=\pi$ for $k=2$ (ball).

Cauchy problem on the half-axis. For $x>0$, $t>0$ and $u(x,0)=\varphi(x)$, applying the Hankel transform, which turns B_k into multiplication by $-\lambda^2$ [4, 5], and inverting it, we obtain the explicit formula

$$u(x,t)=\int_0^\infty G(x,\xi,t)\,\varphi(\xi)\,\xi^k\,d\xi, \qquad G=\frac{(x\xi)^{-\nu}}{2t}\exp\left(-\frac{x^2+\xi^2}{4t}\right) I_\nu\left(\frac{x\xi}{2t}\right), \tag{7}$$

where I_ν is the modified Bessel function. For $k=n-1$, $n\in\mathbb{N}$, formula (7) coincides with the Poisson formula for radially symmetric solutions of the heat equation in $\mathbb{R}^n$; formally setting $k=0$ gives the classical solution on the half-line with an even reflection. This confirms the correctness of (7).

Problem E (elliptic). In the rectangle $D=\{0<x<a,\ 0<y<b\}$ consider the Weinstein equation [3]

$$u_{xx}+u_{yy}+\frac{k}{y}u_y=0, \qquad (x,y)\in D, \tag{8}$$

$$u(0,y)=u(a,y)=0, \qquad u(x,b)=f(x), \qquad y^k u_y(x,y)\to 0 \quad (y\to+0). \tag{9}$$

Let $i_\nu(z)=\Gamma(\nu+1)(2/z)^\nu I_\nu(z)$ be the normalized modified Bessel function: $i_\nu(0)=1$, i_ν increases for $z>0$, and $B_k[i_\nu(\lambda y)]=\lambda^2 i_\nu(\lambda y)$ in the variable y. Then the formal solution of problem E is

$$u(x,y)=\sum_{n=1}^{\infty} f_n \frac{i_\nu(n\pi y/a)}{i_\nu(n\pi b/a)}\sin\frac{n\pi x}{a}, \qquad f_n=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx. \tag{10}$$

Theorem 2. Let $f\in C^1[0,a]$ and $f(0)=f(a)=0$. Then series (10) converges uniformly in $\overline{D}$ and defines the unique classical solution of problem E.

Convergence follows from $0<i_\nu(n\pi y/a)/i_\nu(n\pi b/a)\le 1$ and $\sum|f_n|<\infty$. Furthermore, the asymptotics $i_\nu(n\pi y/a)/i_\nu(n\pi b/a)\sim (b/y)^{\nu+1/2}e^{-n\pi(b-y)/a}$ shows that the solution is infinitely differentiable inside D. Uniqueness follows from the extremum principle: a non-constant solution of (8) attains its maximum and minimum on the part of ∂D where $y>0$, while an extremum on the segment $y=0$ is excluded by the last condition in (9) and an analogue of the Hopf lemma [2, 3].

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Comparison. The structure of the three problems is compared in Table 1.

Table 1. Comparison of the considered problems with the Bessel operator

Feature	Problem P (2)–(3)	Cauchy problem	Problem E (8)–(9)
Type of equation	parabolic	parabolic	elliptic
Singular line	$x=0$	$x=0$	$y=0$
Condition on it	$x^k u_x \to 0$	boundedness	$y^k u_y \to 0$
Basic functions	$x^{-\nu} J_\nu(\mu_n x/l)$	$x^{-\nu} J_\nu(\lambda x)$	$i_\nu(n\pi y/a)\sin\dfrac{n\pi x}{a}$
Tool	Fourier–Bessel series	Hankel transform	Fourier series + i_ν
Uniqueness	energy estimate (4)	maximum principle (bounded solutions)	extremum principle
Representation	series (5)–(6)	integral (7)	series (10)

Remark. The same scheme extends to the multidimensional B-elliptic and B-parabolic equations with the operator $\Delta_B = \sum_{i=1}^{m} B_{k_i,x_i} + \sum_{j=m+1}^{n} \partial^2_{x_j}$ in Kipriyanov spaces [2], and to equations with a fractional time derivative, where the factor $e^{-\mu_n^2 t/l^2}$ is replaced by the Mittag-Leffler function.

CONCLUSION

For the model parabolic and elliptic equations with the Bessel operator B_k, $k>0$, it was shown that the condition $x^k u_x \to 0$ on the singular line is the natural replacement of the classical boundary condition. Uniqueness of solutions was established by the energy method and the extremum principle, and explicit representations were obtained as Fourier–Bessel series and through the Hankel transform with the kernel (7). The obtained formulas contain the classical results for radially symmetric problems as special cases and can be used for numerical computation, for teaching courses on equations of mathematical physics, and as a basis for studying inverse and nonlocal problems for equations with singular coefficients.

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