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CONVERGENCE RATE OF FOURIER–BESSEL SERIES IN THE HEAT CONDUCTION PROBLEM FOR A CYLINDER

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The cooling and heating of cylindrical bodies is a classical problem in heat transfer theory [1, 2], and its analytical solution is expressed as a Fourier–Bessel series. In practical computations the series is truncated to a finite number of terms, so it is important to know how the required number of terms depends on the initial and boundary conditions. This abstract gives a quantitative account of this dependence.

Problem statement. For an infinite cylinder of radius R, we consider the axisymmetric problem in the dimensionless variables $\xi=r/R$, $\tau=a^2t/R^2$:

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial u}{\partial \xi}, \qquad 0\le\xi<1, \qquad u(\xi,0)=f(\xi). \qquad (1)$$

Separation of variables leads to the Bessel equation of order zero; the boundedness condition on the axis rules out the Bessel function of the second kind Y_0, and the solution takes the form

$$u(\xi,\tau)=\sum_k c_k\, J_0(\mu_k\xi)\, e^{-\mu_k^2\tau} \qquad (2)$$

Under the Dirichlet condition $u(1,\tau)=0$, the μ_k are the roots of $J_0(\mu)=0$, whereas under the Neumann condition $u_\xi(1,\tau)=0$ they are the roots of $J_1(\mu)=0$ (including $\mu_0=0$).

Coefficients. For the Dirichlet condition and $f=u_0$ (a uniformly heated cylinder), $c_k=\frac{2u_0}{\mu_k J_1(\mu_k)}$, i.e., the coefficients decay slowly, as $\mu_k^{-1/2}$. In contrast, for the profile $f=u_0(1-\xi^2)$, which is compatible with the boundary condition,

$$c_k=\frac{8u_0}{\mu_k^3 J_1(\mu_k)}, \qquad (3)$$

the coefficients decay rapidly, as $\mu_k^{-5/2}$. Under the Neumann condition, the same profile gives $c_0=u_0/2$ and $c_k=-\frac{4u_0}{\mu_k^2 J_0(\mu_k)}$; as $\tau\to\infty$, the temperature tends to the cross-sectional mean value c_0, which expresses the conservation of energy.

Numerical results. For the axial temperature $\theta(0,\tau)=u(0,\tau)/u_0$ at $\tau=0.05$ with $f=u_0$, one term gives 1.19971 and five terms give 0.98710; for $f=u_0(1-\xi^2)$, the corresponding values are 0.82979 and 0.80038. At $\tau=0.2$, three terms provide an accuracy better than 10^{-6} for both profiles. The results were confirmed by two physical checks: for small τ, the parabolic profile satisfies $\theta(0,\tau)\approx1-4\tau$ (in agreement with the value 0.96000 computed at $\tau=0.01$), and for the Neumann condition at $\tau=0.2$ the values 0.53588 on the axis and 0.48554 on the surface were obtained, showing that the distribution levels out around 1/2.

Conclusion. Using McMahon's asymptotic formula $\mu_k\approx(k-1/4)\pi$, the number of terms required to achieve accuracy ε is estimated by

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$$K \approx \frac{1}{4} + \frac{1}{\pi}\sqrt{\frac{\ln(1/\varepsilon)}{\tau}}$$

At small times, however, the convergence rate depends crucially on whether the initial profile is compatible with the boundary condition: the coefficients decay as $\mu_k^{-5/2}$ for a compatible profile and as $\mu_k^{-1/2}$ for an incompatible one.

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